

# Switched Systems with Multiple Equilibria Under Disturbances: Boundedness and Practical Stability

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**Abstract**—This paper addresses robustness to external disturbances of switched discrete and continuous systems with multiple equilibria. First, we prove that if each subsystem of the switched system is Input-to-State Stable (ISS), then under switching signals that satisfy an average dwell-time bound, the solutions are ultimately bounded within a compact set. The size of this set varies monotonically with the supremum norm of the disturbance signal. These results generalize existing ones in the common equilibrium case to accommodate multiple equilibria. Then, we relax the (global) ISS conditions to consider equilibria that are locally exponentially stable (LES), and we establish practical stability for such switched systems under disturbances. Our motivation for studying this class of switched systems arises from certain motion planning problems in robotics, where primitive movements, each corresponding to an equilibrium point of a dynamical system, must be composed to obtain more complex motions. As a concrete example, we consider the problem of realizing safe adaptive locomotion of a 3D biped under persistent external forcing by switching among motion primitives characterized by LES limit cycles. The results of this paper, however, are relevant to a much broader class of applications, in which composition of different modes of behavior is required to accomplish a task.

**Index Terms**—Switched systems with multiple equilibria; input-to-state stability; practical stability; motion planning.

## I. INTRODUCTION

A switched system is characterized by a family of dynamical systems wherein only one member is active at a time, as governed by a switching signal. From the perspective of control synthesis, switched systems allow stitching individual controllers under a single framework by viewing the dynamics produced by each controller as an individual system. This gives rise to a convenient and modular control strategy that allows the use of pre-designed controllers for generating behaviors richer than what an individual controller is capable of. Owing to these factors, switched systems have been widely used in a broad range of applications—such as power electronics [1], automotive control [2], robotics [3], and air traffic control [4].

A significant amount of research has been directed towards the stability and robustness of switched systems. Stability of switched linear systems was studied in [5] by the construction of a common Lyapunov function which decreases monotonically despite switching. In the absence of a common Lyapunov function, the notion of multiple Lyapunov functions that are

allowed to increase intermittently as long as there is an overall reduction, was proposed in [6]. Instead of dealing with the construction of special classes of Lyapunov functions, [7] proved that the stability properties of the individual subsystems can be translated to the switched system when switching is sufficiently slow in the sense that the switching signal satisfies an average dwell-time bound. The notion of average dwell time was further exploited in [8] to study the input-to-state stability (ISS) of switched continuous systems. Further, [9] and [10] relax the ISS requirement on each subsystem for switching under disturbances. Detailed surveys of results in switched systems can be found in [11]–[13]; it is emphasized, however, that the aforementioned papers consider switching among systems that share a common equilibrium, as does the majority of the switched systems literature.

Various applications demand switching among systems that *do not* share a common equilibrium—such as planning motions of legged [14], [15] and aerial [16] robots, cooperative manipulation among multiple robotic arms [17], power control in multi-cell wireless networks [18], and models for non-spiking of neurons [19]. Such systems are referred to in the literature as *switched systems with multiple equilibria*.<sup>1</sup> To study the behavior of these systems, [18] and [20] established boundedness of the state for switching signals that satisfy an average dwell-time and a dwell-time bound, respectively. The notion of modal dwell-time was introduced in [21], which provided switch-dependent dwell-time bounds, while [22] and [23] established boundedness of solutions via practical stability. The dwell-time bound of [20] was extended to switched discrete systems in [15] and to switched continuous systems with invariant sets in [16]. Yet, papers that deal with multiple equilibria do not study the effect of switching in the presence of disturbances. Conversely, work that considers switching under disturbances is restricted to systems that share a common equilibrium. In the present paper, we address this gap in the literature by studying discrete and continuous switched systems with multiple equilibria under disturbances.

Our interest in switched systems with multiple equilibria stems from their application in certain motion planning and control problems in robotics that require switching among different modes of behavior [24], [25]. As a concrete example, consider dynamically-stable legged robots, in which the ability to switch among a collection of limit-cycle gait primitives enriches the repertoire of robot behaviors. This ability provides the additional flexibility needed for navigating amidst obstacles [14], [15], realizing gait transitions [26], [27], adapting

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<sup>1</sup>To clarify terminology, “switched systems with multiple equilibria” refers to switching among subsystems each of which exhibits a unique equilibrium which may not coincide with the equilibrium of another subsystem.

to external commands [28]–[30], or achieving robustness to disturbances [31], [32]. In this case, each limit-cycle gait primitive corresponds to a distinct equilibrium point of a discrete dynamical system that arises from the corresponding Poincaré map—or forced Poincaré map [33] if disturbances are present. Hence, composing gait primitives can be formulated as a switched discrete system with multiple equilibria, as in [14], [15], [34]–[36]. The present paper provides theoretical tools relevant to ensuring robustness for such systems. It is worth mentioning that these tools can be applied to robust motion planning via the composition of multiple (distinct) equilibrium behaviors for other classes of dynamically-moving robots as well—examples include aerial robots with fixed [37] or flapping [38] wings, snake robots [39], and ballbots [40].

This paper studies the effect of disturbances on switched discrete and continuous systems with multiple (distinct) equilibria. It is proved in Theorems 1 and 2 that if each subsystem has an ISS equilibrium and the switching signal satisfies an average dwell-time constraint, then the solutions of the switched system are ultimately bounded within an explicitly characterized compact set. In addition, motivated by applications, we provide Theorems 3 and 4 that relax the ISS conditions to consider equilibria that are locally exponentially stable (LES) and establish safety guarantees in the form of practical stability<sup>2</sup> under average dwell-time switching signals and disturbances. A notable aspect of these results is that their application does not require explicit knowledge of the disturbances, thus allowing for the design of switching policies that ensure robustness using only local Lyapunov functions. To demonstrate the relevance of these results to practical applications, we consider the problem of safe adaptive locomotion of a 3D biped by switching among limit-cycle gait primitives; this example is representative of a class of tasks in which a system needs to adapt its behavior to (possibly large) variations in the environment within which it operates.

With respect to prior literature, the contribution of this paper is twofold. From a theoretical perspective, it extends current results on boundedness and practical stability of switched systems with multiple equilibria, e.g., [15], [16], [19]–[23], to explicitly consider the effect of disturbances. Furthermore, it naturally generalizes existing ISS results for the common equilibrium case; indeed, when the equilibria of the individual subsystems coalesce, ISS of the switched system can be recovered as a simple consequence of Theorems 1 and 2. From a practical perspective, this paper extends current approaches to motion planning of dynamic robots, by providing safety guarantees for sequentially composing dynamic motion primitives under disturbances. This is unlike existing literature [24], [25], [43], [44], with the only exception being [37], which though requires knowledge of the disturbed dynamics. On the contrary, the results presented here provide safety guarantees in the presence of disturbances using the dynamics in the

<sup>2</sup>In practice, one often encounters systems that do not exhibit equilibria or their equilibria are not stable in the sense of Lyapunov; however, their solutions still evolve in a region where they could “safely” operate, rendering such systems stable in a “practical sense.” This notion of stability is formalized as *practical stability* and has appeared as early as 1961 in LaSalle and Lefschetz’s book [41]; see [42] for further details.

*absence* of disturbances. Preliminary results associated with this work have appeared in [29], [45].

*Notation:*  $\mathbb{R}$  and  $\mathbb{Z}$  denote the real and integer numbers, and  $\mathbb{R}_+$ ,  $\mathbb{Z}_+$  the non-negative reals and integers, respectively. The Euclidean norm is denoted by  $\|\cdot\|$  and  $B_\delta(x) \subset \mathbb{R}^n$  denotes an open-ball (Euclidean) of radius  $\delta > 0$  centered at  $x \in \mathbb{R}^n$ . Let  $\mathcal{A} \subseteq \mathbb{R}^n$ , then  $\overset{\circ}{\mathcal{A}}$  denotes the interior while  $\overline{\mathcal{A}}$  denotes the closure of  $\mathcal{A}$ , respectively. The index  $k \in \mathbb{Z}_+$  represents discrete time. The discrete-time disturbance  $d : \mathbb{Z}_+ \rightarrow \mathbb{R}^m$  is a sequence  $\{d_k\}_{k \in \mathbb{Z}_+}$  with  $d_k \in \mathbb{R}^m$  for  $k \in \mathbb{Z}_+$ . The norm of  $d$  is  $\|d\|_\infty := \sup_{k \in \mathbb{Z}_+} \|d_k\|$ . Let  $t \in \mathbb{R}_+$  represent continuous time. The disturbance  $d : \mathbb{R}_+ \rightarrow \mathbb{R}^m$  that acts in continuous time is assumed to be a piecewise continuous signal with norm  $\|d\|_\infty := \sup_{t \geq 0} \|d(t)\|$ . Abusing notation, we use  $d$ ,  $\|\cdot\|_\infty$  and  $\mathcal{D}$  for both discrete- and continuous-time disturbances. No ambiguity arises as it will always be clear from context whether the signal is discrete or continuous. Finally, a function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is of class  $\mathcal{K}_\infty$  if it is continuous, strictly increasing,  $\alpha(0) = 0$ , and  $\lim_{s \rightarrow \infty} \alpha(s) = \infty$ . A function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is of class  $\mathcal{KL}$  if it is continuous,  $\beta(\cdot, t)$  is of class  $\mathcal{K}_\infty$  for any fixed  $t \geq 0$ ,  $\beta(s, \cdot)$  is strictly decreasing, and  $\lim_{t \rightarrow \infty} \beta(s, t) = 0$  for any fixed  $s \geq 0$ ; see [46].

## II. SWITCHED SYSTEMS WITH MULTIPLE EQUILIBRIA: ISS EQUILIBRIA

This section introduces the classes of discrete and continuous switched systems that are of interest in this work, and provides the main theorems that establish boundedness of solutions under disturbances for sufficiently slow switching.

### A. Switched Discrete Systems

Let  $\mathcal{P}$  be a finite index set and consider the family of discrete-time systems

$$x_{k+1} = f_p(x_k, d_k), \quad p \in \mathcal{P}, \quad (1)$$

where  $x \in \mathbb{R}^n$  is the state and  $d_k \in \mathbb{R}^m$  is the value at time  $k$  of the discrete disturbance signal  $d$ , which belongs to the set of *bounded* disturbances  $\mathcal{D} := \{d : \mathbb{Z}_+ \rightarrow \mathbb{R}^m \mid \|d\|_\infty < \infty\}$ . It is assumed that, for each  $p \in \mathcal{P}$ , the mapping  $f_p : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is continuous in its arguments, and that there exists a unique  $x_p^* \in \mathbb{R}^n$  satisfying  $x_p^* = f_p(x_p^*, 0)$ . Note that the vast majority of the relevant literature assumes that all subsystems  $f_p$  share a common equilibrium point; here, we relax this assumption, and allow for  $x_p^* \neq x_q^*$  when  $p \neq q$ .

To state the main result, we will require each system in the family (1) to be input-to-state stable, as defined below.

**Definition 1** (Adapted from [47]). *The equilibrium point  $x_p^*$  of system  $f_p$  in (1) is input-to-state stable (ISS) if there exists a class  $\mathcal{KL}$  function  $\beta$  and a class  $\mathcal{K}_\infty$  function  $\alpha$  such that for any initial state  $x_0 \in \mathbb{R}^n$  and any bounded input  $d \in \mathcal{D}$ , the solution  $x_k$  exists for all  $k \geq 0$  and satisfies*

$$\|x_k - x_p^*\| \leq \beta(\|x_0 - x_p^*\|, k) + \alpha(\|d\|_\infty). \quad (2)$$

Let  $\sigma : \mathbb{Z}_+ \rightarrow \mathcal{P}$  be a switching signal, mapping the discrete time  $k$  to the index  $\sigma(k) \in \mathcal{P}$  of the subsystem that is active at  $k$ . This gives rise to a discrete switched system of the form

$$x_{k+1} = f_{\sigma(k)}(x_k, d_k). \quad (3)$$

We are interested in establishing boundedness and ultimate boundedness of the solutions of (3) under bounded disturbances, provided that the switching signal is sufficiently “slow on average.” Definition 2 below makes this notion precise.

**Definition 2** (Adapted from [48]). *A switching signal  $\sigma(k)$  has average dwell-time  $N_a > 0$  if the number  $N_\sigma(k, \underline{k}) \in \mathbb{Z}_+$  of switches over any discrete-time interval  $[\underline{k}, k) \cap \mathbb{Z}_+$  where  $k, \underline{k} \in \mathbb{Z}_+$ , satisfies*

$$N_\sigma(k, \underline{k}) \leq N_0 + \frac{k - \underline{k}}{N_a}, \quad \forall k \geq \underline{k} \geq 0 \quad (4)$$

where  $N_0 > 0$  is a finite constant.

We can now state the main result of this section for discrete switched systems, and discuss its consequences.

**Theorem 1.** *Consider the switched system (3) and assume that for each  $p \in \mathcal{P}$  there exists a continuous function  $V_p : \mathbb{R}^n \rightarrow \mathbb{R}_+$  such that for all  $x \in \mathbb{R}^n$  and  $d \in \mathcal{D}$ ,*

$$\underline{\alpha}_p(\|x - x_p^*\|) \leq V_p(x) \leq \bar{\alpha}_p(\|x - x_p^*\|), \quad (5)$$

$$V_p(f_p(x, d)) \leq \lambda_p V_p(x) + \alpha_p(\|d\|_\infty), \quad (6)$$

where  $0 < \lambda_p < 1$  and  $\underline{\alpha}_p, \bar{\alpha}_p, \alpha_p$  are class  $\mathcal{K}_\infty$  functions. Assume further that

$$\limsup_{\|x - x_p^*\| \rightarrow \infty} \frac{V_q(x)}{V_p(x)} < \infty \quad (7)$$

for any  $p, q \in \mathcal{P}$ . Then, there exists  $\bar{N}_a > 0$  so that for any switching signal  $\sigma$  satisfying the average dwell-time constraint (4) with

$$N_0 \geq 1 \text{ and } N_a \geq \bar{N}_a, \quad (8)$$

and for any  $x_0 \in \mathbb{R}^n$ , there exists  $K \in \mathbb{Z}_+$  such that the solution  $\{x_k\}_{k \in \mathbb{Z}_+}$  of (3) satisfies:

(i) for all  $0 \leq k < K$ ,

$$\|x_k - x_{\sigma(k)}^*\| \leq \beta(\|x_0 - x_{\sigma(0)}^*\|, k) + \alpha(\|d\|_\infty) \quad (9)$$

for some  $\beta \in \mathcal{KL}$  and  $\alpha \in \mathcal{K}_\infty$ ;

(ii) for all  $k \geq K$ ,

$$x_k \in \mathcal{M}(\bar{\omega}) := \bigcup_{p \in \mathcal{P}} \{x \in \mathbb{R}^n \mid V_p(x) \leq \bar{\omega}(\|d\|_\infty)\} \quad (10)$$

where

$$\bar{\omega}(\|d\|_\infty) := c + \tilde{\alpha}(\|d\|_\infty), \quad (11)$$

for some  $c > 0$  and  $\tilde{\alpha} \in \mathcal{K}_\infty$ .

Proving Theorem 1 is postponed until Section V-A. Explicit expressions for the bound  $\bar{N}_a$  on the average dwell-time  $N_a$ , and for  $\bar{\omega}$  in the characterization of the set  $\mathcal{M}(\bar{\omega})$  are important in applications, and are given before proofs, in Section III-B.

Let us now briefly discuss some aspects of Theorem 1. First, note that by [47, Definition 3.2], conditions (5)-(6) imply that  $V_p$  is an ISS-Lyapunov function for the  $p$ -th subsystem, which by [47, Lemma 3.5], entails that the 0-input fixed point of  $f_p$  is ISS. Since we are interested in switching among systems from the family (1), condition (7) is added to ensure that the ratio of the Lyapunov functions corresponding to the subsystems involved in switching is bounded. As will be shown in

Section III-A below, (7) essentially implies the existence of a finite  $\mu > 0$  such that  $V_q(x) \leq \mu V_p(x)$  for  $p, q \in \mathcal{P}$  over the domain of interest, a condition which is common in switched systems literature; see [11, equation (3.6)], [7, equation (8)], or [8, equation (9)], for example. Now, with conditions (5)-(6) and (7) in place, Theorem 1 states that if each system in the family (1) is ISS, the solution of (3) is uniformly<sup>3</sup> bounded and uniformly ultimately bounded within the compact set  $\mathcal{M}(\bar{\omega})$  characterized by (10). Furthermore, (10) indicates that the “size” of  $\mathcal{M}(\bar{\omega})$  reduces proportionally with the norm  $\|d\|_\infty$  of the disturbance. Note, however, that  $\mathcal{M}(\bar{\omega})$  does *not* collapse to a point when the disturbance signal  $d$  vanishes; indeed, if  $d = 0$ , (11) implies that  $\bar{\omega}(0) = c > 0$  and the solutions of (3) are ultimately bounded to the 0-input compact set  $\mathcal{M}(c)$  that contains the equilibria  $x_p^* \in \mathcal{M}(c)$ ,  $\forall p \in \mathcal{P}$ .

It should be noted that Theorem 1 does not establish ISS for (3) with respect to the compact set  $\mathcal{M}(c)$ , because  $\mathcal{M}(c)$  is not positively invariant<sup>4</sup> under the 0-input dynamics of (3). In fact, for suitable switching signals satisfying the requirements of the theorem, solutions of (3) can be found that start within  $\mathcal{M}(c)$  and—while evolving in the absence of the disturbance—escape from  $\mathcal{M}(c)$  before they return to  $\mathcal{M}(c)$  and be trapped forever in it; see also Remark 2 in Section V-A below for how this behavior can emerge. Note also that although the estimate (9) is reminiscent of (2) in Definition 1, it extends only up to a finite integer  $K$ , and it does not represent point-to-set distance from  $\mathcal{M}(c)$  as establishing set-ISS for (3) would require [49]. However, when all the subsystems in the family (1) share the same equilibrium, then ISS can be recovered, as the following corollary shows. Corollary 1 provides the counterpart of [8, Theorem 3.1] for discrete switched systems.

**Corollary 1.** *Consider (3) with  $x_p^* = 0$  for all  $p \in \mathcal{P}$ . Let the assumptions of Theorem 1 hold, and further assume that*

$$\limsup_{\|x\| \rightarrow 0} \frac{V_q(x)}{V_p(x)} < \infty \quad (12)$$

for all  $p, q \in \mathcal{P}$ . Then, the system (3) is ISS.

While the set  $\mathcal{M}(\bar{\omega})$  in Theorem 1 is not positively invariant, one can identify a (compact) subset  $\Omega_1$  of initial conditions in  $\mathcal{M}(c)$  such that the corresponding solutions never leave  $\Omega_2 = \mathcal{M}(\bar{\omega})$ . This property corresponds to the notion of *practical stability* with respect to the sets  $\Omega_1$  and  $\Omega_2$  [22], [42], and is made precise by the following definition and corollary.

**Definition 3** (Adapted from [42]). *The switched system (3) is practically stable for a disturbance  $d \in \mathcal{D}$  with respect to the sets  $\Omega_1$  and  $\Omega_2$ , if  $x_0 \in \Omega_1$  implies  $x_k \in \Omega_2$  for all  $k \in \mathbb{Z}_+$ .*

**Corollary 2.** *Under the assumptions of Theorem 1, there exists a compact set  $\Omega_1 \subset \mathcal{M}(c)$  with  $x_p^* \in \overset{\circ}{\Omega}_1$  for all  $p \in \mathcal{P}$ , such that the switched system (3) is practically stable for any  $d \in \mathcal{D}$  with respect to the sets  $\Omega_1$  and  $\Omega_2 := \mathcal{M}(\bar{\omega})$ .*

<sup>3</sup>The term “uniformly” refers to uniformity over the set of switching signals that satisfy (4) for  $N_0 \geq 1$  and  $N_a \geq \bar{N}_a$ , as required by (8) of Theorem 1.

<sup>4</sup>In the terminology of [49],  $\mathcal{M}(\bar{\omega})$  is not a 0-invariant set for (3).

## B. Switched Continuous Systems

As in Section II-A, let  $\mathcal{P}$  be a finite index set and consider the family of continuous-time systems

$$\dot{x}(t) = f_p(x(t), d(t)), \quad p \in \mathcal{P}, \quad (13)$$

where  $x \in \mathbb{R}^n$  is the state of the system and  $d(t) \in \mathbb{R}^m$  is the value of the continuous-time disturbance signal  $d$  at time  $t$  which belongs to the set of *bounded* disturbances  $\mathcal{D} := \{d : \mathbb{R}_+ \rightarrow \mathbb{R}^m \mid \|d\|_\infty < \infty, d \text{ piecewise continuous}\}$ . It is assumed that, for each  $p \in \mathcal{P}$ , the vector field  $f_p : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is locally Lipschitz in its arguments, and that there exists a unique  $x_p^* \in \mathbb{R}^n$  with  $0 = f_p(x_p^*, 0)$ . As in Section II-A, we allow for  $x_p^* \neq x_q^*$  when  $p \neq q$ .

Analogous to Section II-A, we will require each system in the family (13) to be input-to-state stable, as defined below.

**Definition 4** (Adapted from [49]). *The equilibrium point  $x_p^*$  of system  $f_p$  in (13) is ISS if there exist a class  $\mathcal{KL}$  function  $\beta$  and a class  $\mathcal{K}_\infty$  function  $\alpha$  such that for any initial state  $x(0) \in \mathbb{R}^n$  and any bounded input  $d \in \mathcal{D}$ , the solution  $x(t)$  exists for all  $t \geq 0$  and satisfies*

$$\|x(t) - x_p^*\| \leq \beta(\|x(0) - x_p^*\|, t) + \alpha(\|d\|_\infty). \quad (14)$$

Let  $\sigma : \mathbb{R}_+ \rightarrow \mathcal{P}$  be a switching signal mapping the time instant  $t$  to the index  $\sigma(t) \in \mathcal{P}$  of the subsystem that is active at  $t$ . It is assumed that  $\sigma(t)$  is right-continuous. The switching signal gives rise to the continuous-time switched system

$$\dot{x}(t) = f_{\sigma(t)}(x(t), d(t)). \quad (15)$$

The solution  $x(t) := \phi(t, x(0), \sigma(t), d(t))$  of (15) is a sequential concatenation of each subsystem's solution as governed by the switching signal. Let  $\{t_n\}_{n \in \mathbb{Z}_+}$  with  $t_n \in \mathbb{R}_+$  be a strictly monotonically increasing sequence of switching times. Clearly, continuity of  $f_p(x, d)$  and piecewise continuity of  $d(t)$  imply that  $x(t)$  is continuous over  $(t_n, t_{n+1})$ , i.e., between subsequent switches. Furthermore, for any  $t_n$ , the subsystem  $f_{\sigma(t_n)}$  that is switched in and is active over  $[t_n, t_{n+1})$  is initialized by  $x(t_n) = \lim_{t \nearrow t_n} x(t)$  ensuring that  $x(t)$  is continuous at  $t_n$ . Hence,  $x(t)$  is continuous for all  $t \geq 0$ .

As in the discrete-time case, the main result of this section is stated for switching signals  $\sigma$  with sufficiently slow switching on average; the following definition formalizes this notion.

**Definition 5** (Adapted from [7]). *A switching signal  $\sigma(t)$  has average dwell-time  $N_a > 0$  if the number  $N_\sigma(t, \underline{t}) \in \mathbb{Z}_+$  of switches over any interval  $[\underline{t}, t) \subset \mathbb{R}_+$  satisfies*

$$N_\sigma(t, \underline{t}) \leq N_0 + \frac{t - \underline{t}}{N_a}, \quad \forall t \geq \underline{t} \geq 0 \quad (16)$$

where  $N_0 > 0$  is a finite constant.

We are now ready to state the main result of this section for switched continuous systems.

**Theorem 2.** *Consider the switched system (15) and assume that for each  $p \in \mathcal{P}$  there exists a continuously differentiable function  $V_p : \mathbb{R}^n \rightarrow \mathbb{R}_+$  such that for all  $x \in \mathbb{R}^n$  and  $d \in \mathcal{D}$ ,*

$$\underline{\alpha}_p(\|x - x_p^*\|) \leq V_p(x) \leq \bar{\alpha}_p(\|x - x_p^*\|), \quad (17)$$

$$\frac{\partial V_p}{\partial x} f_p(x, d) \leq -\lambda_p V_p(x) + \alpha_p(\|d\|_\infty), \quad (18)$$

where  $\lambda_p > 0$  and  $\underline{\alpha}_p, \bar{\alpha}_p, \alpha_p$  are class  $\mathcal{K}_\infty$  functions. Assume further that

$$\limsup_{\|x - x_p^*\| \rightarrow \infty} \frac{V_q(x)}{V_p(x)} < \infty \quad (19)$$

for  $p, q \in \mathcal{P}$ . Then, there exists  $\bar{N}_a > 0$  so that for any switching signal  $\sigma$  satisfying the average dwell-time constraint (16) with

$$N_0 \geq 1 \text{ and } N_a \geq \bar{N}_a, \quad (20)$$

and for any  $x(0) \in \mathbb{R}^n$ , there exists  $T \in \mathbb{R}_+$  such that the solution  $x(t) := \phi(t, x(0), \sigma(t), d(t))$  of (15) satisfies:

(i) for all  $0 \leq t < T$ ,

$$\|x(t) - x_{\sigma(t)}^*\| \leq \beta(\|x(0) - x_{\sigma(0)}^*\|, t) + \alpha(\|d\|_\infty) \quad (21)$$

for some  $\beta \in \mathcal{KL}$ ,  $\alpha \in \mathcal{K}_\infty$ ;

(ii) for all  $t \geq T$ ,

$$x(t) \in \mathcal{M}(\bar{\omega}) := \bigcup_{p \in \mathcal{P}} \{x \in \mathbb{R}^n \mid V_p(x) \leq \bar{\omega}(\|d\|_\infty)\} \quad (22)$$

where

$$\bar{\omega}(\|d\|_\infty) := c + \tilde{\alpha}(\|d\|_\infty) \quad (23)$$

for some  $c > 0$  and  $\tilde{\alpha} \in \mathcal{K}_\infty$ .

A proof of Theorem 2 is presented in Section V-B below, and explicit expressions for  $\bar{N}_a$  in (20) and  $\bar{\omega}$  in (22) are provided in Section III-C. It is only mentioned here that Theorem 2 is completely analogous to Theorem 1, establishing uniform boundedness by (21) and uniform ultimate boundedness in the compact set  $\mathcal{M}(\bar{\omega})$  characterized by (22) of the solutions of (15). Theorem 2 does not establish ISS stability of (15) with respect to the compact set  $\mathcal{M}(c)$ , since this set is not invariant under the 0-input dynamics of (15); see Example 2 in [50, Section V-B] for an illustration of this behavior. However, as was the case with Corollary 1, ISS can be recovered when the equilibria of all subsystems coalesce. The following corollary makes this statement precise by particularizing Theorem 2 to the common equilibrium case, and it shows that [8, Theorem 3.1] can be obtained as a special case of Theorem 2.

**Corollary 3.** *Consider (15) with  $x_p^* = 0$  for all  $p \in \mathcal{P}$ . Let the assumptions of Theorem 2 hold, and further assume that*

$$\limsup_{\|x\| \rightarrow 0} \frac{V_q(x)}{V_p(x)} < \infty \quad (24)$$

for all  $p, q \in \mathcal{P}$ . Then, the system (15) is ISS.

Finally, Definition 6 and Corollary 4 below are the counterparts of Definition 3 and Corollary 2 for the switched system (15), with Corollary 4 establishing the existence of sets  $\Omega_1$  and  $\Omega_2$  with respect to which the switched system with multiple equilibria (15) is practically stable.

**Definition 6** (Adapted from [42]). *The switched system (15) is practically stable for a disturbance  $d \in \mathcal{D}$  with respect to the sets  $\Omega_1$  and  $\Omega_2$ , if  $x(0) \in \Omega_1$  implies  $x(t) \in \Omega_2$  for all  $t \geq 0$ .*

**Corollary 4.** *Under the assumptions of Theorem 2, there exists a compact set  $\Omega_1 \subset \mathcal{M}(c)$  with  $x_p^* \in \mathring{\Omega}_1$  for all  $p \in \mathcal{P}$ , such that the switched system (15) is practically stable for any  $d \in \mathcal{D}$  with respect to the sets  $\Omega_1$  and  $\Omega_2 := \mathcal{M}(\bar{\omega})$ .*

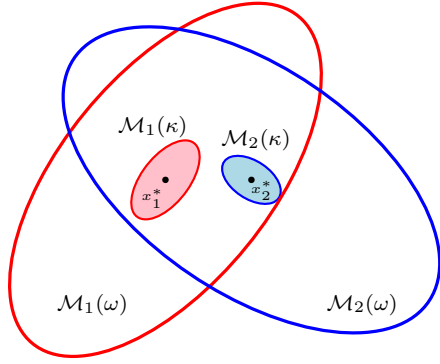


Fig. 1. Illustration of the set construction. The sublevel sets for system 1 are in red and the sublevel sets of system 2 are in blue; the construction leads to  $\mathcal{M}_1(\kappa) \cup \mathcal{M}_2(\kappa) \subseteq \mathcal{M}_1(\omega) \cap \mathcal{M}_2(\omega)$  as Remark 1 explains.

### III. SET CONSTRUCTIONS AND EXPLICIT BOUNDS

This section characterizes the family of switching signals required by Theorems 1 and 2 by providing explicit expressions for the dwell-time bound  $\bar{N}_a$  in (8) and (20), respectively. Explicit expressions of  $\bar{\omega}$  are also given, thereby determining the sets  $\mathcal{M}(\bar{\omega})$  within which the solutions ultimately converge. We begin with relevant set constructions motivated by [20].

#### A. Set Constructions

Suppose that  $V_p$  is a function satisfying (5)-(6) in the case of discrete or (17)-(18) in the case of switched continuous systems. The  $\kappa$ -sublevel set of  $V_p$  is defined as

$$\mathcal{M}_p(\kappa) := \{x \in \mathbb{R}^n \mid V_p(x) \leq \kappa\},$$

and the union of the sublevel sets over  $\mathcal{P}$  is denoted as

$$\mathcal{M}(\kappa) := \bigcup_{p \in \mathcal{P}} \mathcal{M}_p(\kappa); \quad (25)$$

see Fig. 1. Next, we define a positive constant

$$\omega(\kappa) := \max_{p \in \mathcal{P}} \max_{x \in \mathcal{M}(\kappa)} V_p(x), \quad (26)$$

which is well defined since  $\mathcal{M}(\kappa)$  is compact for any  $\kappa > 0$  and  $\mathcal{P}$  is finite. Intuitively, the definition of  $\omega(\kappa)$  by (26) enlarges each sublevel set  $\mathcal{M}_p(\kappa)$  so that the resulting enlarged set  $\mathcal{M}_p(\omega(\kappa))$  includes the sets  $\mathcal{M}_q(\kappa)$  for all  $q \in \mathcal{P}$ . An illustration of this construction can be seen in Fig. 1 and the following remark makes this intuition precise.

**Remark 1.** By the definition (26) of  $\omega(\kappa)$ ,  $V_p(x) \leq \omega(\kappa)$  for any  $x \in \mathcal{M}(\kappa)$  and any  $p \in \mathcal{P}$ . Thus,  $\mathcal{M}(\kappa) \subseteq \mathcal{M}_p(\omega(\kappa))$  for all  $p \in \mathcal{P}$ , implying that  $\mathcal{M}(\kappa) \subseteq \bigcap_{p \in \mathcal{P}} \mathcal{M}_p(\omega(\kappa))$ .

We now establish a relationship between the values  $V_p(x)$  and  $V_q(x)$  of any pair  $V_p, V_q$  of ISS-Lyapunov functions at a given point  $x \in \mathbb{R}^n$  as the system switches between the corresponding two subsystems  $p, q \in \mathcal{P}$ ,  $p \neq q$ . Consider the ratio  $V_q(x)/V_p(x)$ , and let

$$\mu_p(\kappa) := \max_{q \in \mathcal{P}} \sup_{x \notin \mathring{\mathcal{M}}_p(\kappa)} \frac{V_q(x)}{V_p(x)}, \quad (27)$$

which is bounded<sup>5</sup> due to (7) and (19). This constant provides a bound on how much the value of the Lyapunov function can change on switching. Clearly,

$$\forall p, q \in \mathcal{P}, \quad V_q(x) \leq \mu_p(\kappa) V_p(x) \quad \forall x \notin \mathring{\mathcal{M}}_p(\kappa).$$

To make this bound independent of  $p$ , let

$$\mu(\kappa) := \max_{p \in \mathcal{P}} \mu_p(\kappa), \quad (28)$$

which implies that

$$\forall p, q \in \mathcal{P}, \quad V_q(x) \leq \mu(\kappa) V_p(x) \quad \forall x \notin \mathring{\mathcal{M}}_p(\kappa). \quad (29)$$

Due to the interchangeability of the indices  $p$  and  $q$ , it also holds that  $V_p(x) \leq \mu(\kappa) V_q(x)$  as long as  $x \notin \mathring{\mathcal{M}}_q(\kappa)$ . Hence, when  $x \notin \mathring{\mathcal{M}}_p(\kappa) \cup \mathring{\mathcal{M}}_q(\kappa)$ , we can write  $V_q(x) \leq \mu(\kappa)^2 V_p(x)$ , from which it follows that

$$\mu(\kappa) \geq 1, \quad (30)$$

since  $V_q$  is positive definite for  $x \notin \mathring{\mathcal{M}}_p(\kappa) \cup \mathring{\mathcal{M}}_q(\kappa)$ . Finally, in the context of the switched system (3), it is worth noting that (29) holds for a switching instant  $k_n$  even if  $x_{k_n} \in \mathring{\mathcal{M}}(\kappa)$ , as long as  $x_{k_n} \notin \mathring{\mathcal{M}}_{\sigma(k_n-1)}(\kappa)$ . A similar statement can be made for the switched continuous system (15).

Given the parameter  $\kappa$ , computation of  $\mu(\kappa)$  and  $\omega(\kappa)$  can be challenging—numerical computations based on discretizing the state-space become impractical as the dimension of the system grows. This challenge has been pointed out in [20], where the authors highlighted the need for efficient tools for computing  $\mu(\kappa)$  and  $\omega(\kappa)$ . To alleviate this issue, the following proposition provides analytical bounds for  $\mu(\kappa)$  and  $\omega(\kappa)$  in the case where  $V_p$  are quadratic functions.

**Proposition 1.** Let  $V_p(x) = (x - x_p^*)^T S_p (x - x_p^*)$  for all  $p \in \mathcal{P}$  be a family of positive definite quadratic functions and  $\lambda_{\min}(S_p)$  be the minimum and  $\lambda_{\max}(S_p)$  be the maximum eigenvalues of  $S_p$ , respectively. Given  $\kappa > 0$ , define  $\omega(\kappa)$  by (26) and  $\mu(\kappa)$  by (28). Then, the following hold

$$\omega(\kappa) \leq \max_{p, q \in \mathcal{P}} \left( \lambda_{\max}(S_p) \left( \sqrt{\frac{\kappa}{\lambda_{\min}(S_q)}} + \|x_p^* - x_q^*\| \right)^2 \right) \quad (31)$$

$$\mu(\kappa) \leq \max_{p, q \in \mathcal{P}} \left( \frac{\lambda_{\max}(S_q)}{\lambda_{\min}(S_p)} \left( 1 + \sqrt{\frac{\lambda_{\max}(S_p)}{\kappa}} \|x_p^* - x_q^*\| \right)^2 \right). \quad (32)$$

The proof of Proposition 1 is provided in Appendix A. Note that if  $\underline{\alpha}_p(\cdot)$  and  $\bar{\alpha}_p(\cdot)$  in Theorems 1 and 2 are available for all  $p \in \mathcal{P}$ , then, following steps similar to those in the proof of Proposition 1, analytical bounds for  $\omega(\kappa)$  can be obtained for these non-quadratic Lyapunov functions as well; however, we cannot obtain a general analytical bound for  $\mu(\kappa)$ . Note that the bound (31) for  $\omega(\kappa)$  has also been obtained in [19].

The aforementioned set constructions allow us to provide explicit expressions for the average dwell-time bound  $\bar{N}_a$  in

<sup>5</sup>To see this, let  $a_1 := \limsup_{\|x - x_p^*\| \rightarrow \infty} V_q(x)/V_p(x)$  which is bounded by (7) and (19). Hence, there exists an  $r > 0$  such that for any  $x$  with  $\|x - x_p^*\| > r$ , we have  $V_q(x)/V_p(x) \leq a_1 + 1$ . Expand  $r$  if necessary to ensure that  $\mathcal{M}(\kappa) \subset B_r(x_p^*)$ . Note that  $V_q(x)/V_p(x)$  is continuous on  $\mathbb{R}^n \setminus \{x_p^*\}$  hence it is also continuous on  $\bar{B}_r(x_p^*) \setminus \mathring{\mathcal{M}}_p(\kappa) \subset \mathbb{R}^n \setminus \{x_p^*\}$  which is compact. Then, there exists  $a_2 > 0$  such that  $V_q(x)/V_p(x) < a_2$  for any  $x \in \bar{B}_r(x_p^*) \setminus \mathring{\mathcal{M}}_p(\kappa)$ . Therefore  $V_q(x)/V_p(x) < \max\{a_1 + 1, a_2\}$  for all  $x \notin \mathring{\mathcal{M}}_p(\kappa)$ , ensuring the boundedness of  $\mu_p(\kappa)$  in (27).

(8) and (20), and for the characterization of the sets  $\mathcal{M}(\bar{\omega})$  in (10) and (22) of Theorems 1 and 2, respectively. Although these expressions are derived in the proofs of Section V below, we provide them here to ease their use in applications.

### B. Switched Discrete Systems: Explicit Bounds

For the sake of notational convenience in (6), let  $\lambda := \max_{p \in \mathcal{P}} \lambda_p$  and  $\hat{\alpha}(\|d\|_\infty) := \max_{p \in \mathcal{P}} \{\alpha_p(\|d\|_\infty)\}$ . Then, for any  $p \in \mathcal{P}$  we can write (6) as

$$V_p(x_{k+1}) \leq \lambda V_p(x_k) + \hat{\alpha}(\|d\|_\infty) , \quad (33)$$

where  $\lambda \in (0, 1)$  and  $\hat{\alpha}$  are independent of  $p$ .

Let  $\epsilon$  be any constant that lies within  $(\lambda, 1)$ . Then, it will be shown in the proof of Theorem 1 that the lower bound on the average dwell-time, i.e.,  $\bar{N}_a$  in (8), is

$$\bar{N}_a = \frac{\ln \mu(\kappa)}{\ln(\epsilon/\lambda)} , \quad (34)$$

where the constant  $\mu(\kappa) \geq 1$  is defined by (28). Furthermore, the compact set  $\mathcal{M}(\bar{\omega})$  in Theorem 1(ii) is characterized by

$$\bar{\omega}(\|d\|_\infty) := \mu(\kappa)^{N_0} \omega(\kappa) + \frac{\mu(\kappa)^{N_0}}{1 - \epsilon} \hat{\alpha}(\|d\|_\infty) , \quad (35)$$

with  $\omega$  given by (26). From (35) the constant  $c > 0$  and the function  $\tilde{\alpha} \in \mathcal{K}_\infty$  participating in (11) can be readily identified.

It is remarked that the constants  $\epsilon$  and  $\kappa$  are design parameters available for tuning the frequency of the switching signal. The choice of  $\epsilon \in (\lambda, 1)$  provides a tradeoff between robustness of the switched system (3) and the switching frequency. In more detail, if  $\epsilon$  is chosen close to  $\lambda$ , the lower bound on the average dwell-time (34) becomes large, thus limiting the number of switches  $N_\sigma(k, \underline{k})$  in any time interval  $[\underline{k}, k]$ , as (4) implies. On the other hand, if  $\epsilon$  is chosen close to 1, the size of the compact set  $\mathcal{M}(\bar{\omega})$  to which the solutions ultimately converge increases as  $\bar{\omega}$  in (35) increases. As a result, the solutions of (3) are permitted to wander in a larger set, indicating low robustness of (3) to disturbances. Hence, slower switching signals result in tighter trapping regions.

The effect of  $\kappa$  is more involved. Picking smaller values of  $\kappa$  results in inner sublevel level sets of  $V_p$ , thus reducing the size of  $\mathcal{M}(\kappa)$  in (25) and causing  $\mu_p(\kappa)$  to increase as the supremum in (27) increases. Hence, smaller values of  $\kappa$  result in larger values of  $\mu(\kappa)$  by (28), leading to slower switching frequencies as well as larger compact attractive sets  $\mathcal{M}(\bar{\omega})$ ; observe the role of  $\mu(\kappa)$  in (34) and (35). On the other hand, picking a larger  $\kappa$  will result in smaller  $\mu(\kappa)$  allowing for faster switches, however,  $\omega(\kappa)$  also increases making its effect on  $\bar{\omega}$  unclear; observe the  $\mu^{N_0} \omega(\kappa)$  term in (35).

### C. Switched Continuous Systems: Explicit Bounds

To simplify notation in (18), we define  $\lambda := \min_{p \in \mathcal{P}} \lambda_p > 0$  and  $\hat{\alpha}(\|d\|_\infty) := \max_{p \in \mathcal{P}} \{\alpha_p(\|d\|_\infty)\}$ . Then, for any  $p \in \mathcal{P}$ ,

$$\frac{\partial V_p}{\partial x} f_p(x, d) \leq -\lambda V_p(x) + \hat{\alpha}(\|d\|_\infty) , \quad (36)$$

where  $\lambda > 0$  and  $\hat{\alpha}(\|d\|_\infty)$  are independent of  $p$ .

Let  $\epsilon$  be any constant in the open interval  $(0, \lambda)$ , then the lower bound on the average dwell time  $\bar{N}_a$  in Theorem 2 is

$$\bar{N}_a = \frac{\ln \mu(\kappa)}{\lambda - \epsilon} , \quad (37)$$

where  $\mu(\kappa) \geq 1$  is defined by (28). The compact set  $\mathcal{M}(\bar{\omega})$  in Theorem 2(ii), within which solutions of (15) ultimately become trapped corresponds to

$$\bar{\omega}(\|d\|_\infty) := \mu(\kappa)^{1+N_0} \omega(\kappa) + \mu(\kappa)^{1+N_0} \frac{1}{\epsilon} \hat{\alpha}(\|d\|_\infty) , \quad (38)$$

from which the constant  $c > 0$  and the class- $\mathcal{K}_\infty$  function  $\tilde{\alpha}$  in (23) can be easily recognized.

As in the case of the discrete switched systems, the constant  $\epsilon \in (0, \lambda)$  presents a tradeoff between the robustness of the system and the switching frequency; setting  $\epsilon$  close to 0 increases the disturbance term in (38) while setting  $\epsilon$  close to  $\lambda$  increases  $\bar{N}_a$  in (37). Regarding the effect of  $\kappa$ , the discussion is identical to that in Section III-B.

## IV. SWITCHED SYSTEMS WITH MULTIPLE EQUILIBRIA: LES EQUILIBRIA

Applying Theorems 1 and 2 require each subsystem of (3) and (15), respectively, to be globally ISS. However, various applications call for switching among systems with only local stability properties. Hence, in this section, we relax the global ISS requirement to mere LES, and establish practical stability for (3) and (15) according to Definitions 3 and 6, provided that an average dwell-time constraint is satisfied by switching.

### A. Switched Discrete Systems

Consider a finite family of subsystems indexed by  $p \in \mathcal{P}$ , such that each subsystem  $f_p : \mathcal{X}_p \times \mathbb{R}^m \rightarrow \mathcal{X}_p$  where  $\mathcal{X}_p$  is an open subset of  $\mathbb{R}^n$ . Here, we require  $f_p$  to be locally Lipschitz in its arguments. For each  $p \in \mathcal{P}$ , let  $x_p^* \in \mathcal{X}_p$  be a 0-input equilibrium point of (1) and assume that there exists a locally Lipschitz function  $V_p : \mathcal{X}_p \rightarrow \mathbb{R}_+$  which, for all  $x \in \mathcal{X}_p$ , satisfies (5) for suitable class- $\mathcal{K}_\infty$  functions<sup>6</sup>  $\underline{\alpha}_p$  and  $\bar{\alpha}_p$ , and

$$V_p(f_p(x, 0)) \leq \lambda_p V_p(x) , \quad (39)$$

where  $\lambda_p \in (0, 1)$ .

Now, let  $\bar{\kappa}_p > 0$  be such that

$$\mathcal{M}_p(\bar{\kappa}_p) := \{x \in \mathbb{R}^n \mid V_p(x) \leq \bar{\kappa}_p\} \subset \mathcal{X}_p . \quad (40)$$

For the sake of convenience, define

$$\mathcal{X} := \bigcap_{p \in \mathcal{P}} \overset{\circ}{\mathcal{M}}_p(\bar{\kappa}_p) , \quad (41)$$

which is an open subset of  $\mathbb{R}^n$ ; see Fig. 2 for an illustration of this set. In contrast to the global case of Section II, here we will require the following *composability* assumption, i.e.,

$$x_p^* \in \mathcal{X} , \quad \forall p \in \mathcal{P} , \quad (42)$$

which ensures the non-emptiness of  $\mathcal{X}$  and the feasibility of switching among the members of the family of subsystems.

<sup>6</sup>To avoid confusion, note that these functions may differ from those in (5).

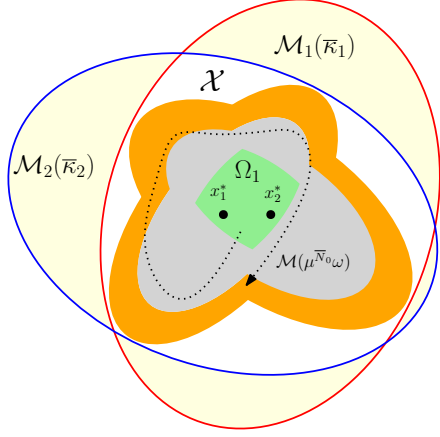


Fig. 2. Illustration of Theorem 3. The green set is  $\Omega_1$ , the grey set is  $\mathcal{M}(\mu^{\bar{N}_0}\omega)$ , and the orange set is  $\mathcal{M}(\bar{\omega}(\delta))$ . The black dots represent a solution of the switched system which satisfies Theorem 3, under disturbances  $d \in \mathcal{D}$  with  $\|d\|_\infty < \delta$ .

We use the definitions and set constructions developed in Section III-A with the local restriction that requires all sublevel sets to lie within the domain  $\mathcal{X}$  defined by (41); see Fig. 2. Additionally, we modify the definition of  $\mu$  from (27), (28) to,

$$\mu(\kappa) := \max_{p,q \in \mathcal{P}} \sup_{x \in \mathcal{X} \setminus \mathcal{M}_p(\kappa)} \frac{V_q(x)}{V_p(x)}, \quad (43)$$

to incorporate the local restriction to  $\mathcal{X}$ . We are now ready to present the first main result of this section; refer to Fig. 2 for the associated set constructions.

**Theorem 3.** *Consider the switched system (3) and assume that for each  $p \in \mathcal{P}$  there exists a locally Lipschitz function  $V_p : \mathcal{X}_p \rightarrow \mathbb{R}_+$  which satisfies (5) for suitable class- $\mathcal{K}_\infty$  functions  $\underline{\alpha}_p$  and  $\bar{\alpha}_p$  and (39) for some  $0 < \lambda_p < 1$ . Suppose further that there exist  $\kappa > 0$  and  $\bar{N}_0 \geq 1$  such that*

$$\mathcal{M}(\mu(\kappa)^{\bar{N}_0}\omega(\kappa)) \subset \mathcal{X}, \quad (44)$$

where  $\mu$  and  $\omega$  are given by (43) and (26), respectively,  $\mathcal{M}$  is the union over  $\mathcal{P}$  of the corresponding sublevel sets of  $V_p$ , and  $\mathcal{X}$  is the domain (41). Then, there exists  $\delta > 0$  such that for any disturbance signal  $d \in \mathcal{D}$  with  $\|d\|_\infty < \delta$ , and for any switching signal  $\sigma$  satisfying (4) with

$$\bar{N}_0 \geq N_0 \geq 1 \text{ and } N_a \geq \bar{N}_a, \quad (45)$$

where  $\bar{N}_0$  satisfies (44) with the selected  $\kappa > 0$  and  $\bar{N}_a$  is defined by (34), the switched system (3) is practically stable with respect to the compact sets  $\Omega_1$  and  $\Omega_2$  defined by

$$\Omega_1 := \bigcap_{p \in \mathcal{P}} \mathcal{M}_p(\omega(\kappa)) \text{ and } \Omega_2 := \bigcup_{p \in \mathcal{P}} \mathcal{M}_p(\bar{\omega}(\|d\|_\infty)),$$

where

$$\bar{\omega}(\|d\|_\infty) := \mu(\kappa)^{N_0}\omega(\kappa) + \tilde{\alpha}(\|d\|_\infty), \quad (46)$$

for a suitable class- $\mathcal{K}_\infty$  function  $\tilde{\alpha}$ .

A proof of Theorem 3 will be presented in Section V-C.

We now discuss some aspects that are relevant to the implementation of Theorem 3. First, note that, unlike the global ISS requirement of Theorem 1, Theorem 3 only requires

that for each  $p \in \mathcal{P}$  a local Lyapunov function  $V_p$  satisfying (5) and (39) is available. Then, the sets  $\mathcal{M}_p(\bar{\kappa}_p)$  in (40) represent compact inner approximations of the basins of attraction of the equilibrium points  $x_p^*$  for  $p \in \mathcal{P}$ . Although generally it may be challenging to obtain such approximations, in many practical situations semi-definite programming tools, such as sum-of-squares (SoS) [51], can be used to facilitate this task and obtain  $\bar{\kappa}_p > 0$  for which  $\mathcal{M}_p(\bar{\kappa}_p)$  satisfies (40).

Let us now turn our attention to the verification of condition (44) and provide a trial-and-error procedure to obtain suitable  $\kappa$  and  $\bar{N}_0$ . First, choose a  $\kappa > 0$  and compute  $\omega(\kappa)$  and  $\mu(\kappa)$  by (26) and (43), respectively; in the case of quadratic Lyapunov functions, this computation can be assisted by Proposition 1. Then, if a  $\bar{N}_0 \geq 1$  can be found for which (44) holds, continue with the implementation of the theorem and compute  $\bar{N}_a$  using (34). However, if such  $\bar{N}_0$  cannot be found, repeat the above procedure with a new choice of  $\kappa > 0$ . Section VI provides a concrete example for choosing suitable  $\kappa$  and  $\bar{N}_0$ . Note though that this procedure requires checking the set inclusion (44), a task that can be burdensome in general cases. However, the procedure can be greatly simplified in the case of quadratic Lyapunov functions by using readily available convex optimization tools as in [52, Section 8.4].

As a final comment, note that Theorem 3 does not require any specific information about the disturbance  $d \in \mathcal{D}$  besides the fact that  $f_p$  is locally Lipschitz with respect to  $d_k$ . In addition, computing  $\bar{N}_a$  by (34) also does not require explicit knowledge of the disturbance. Hence, Theorem 3 provides a disturbance-agnostic method to design switching signals for switching among subsystems that exhibit LES equilibria.

## B. Switched Continuous Systems

Switching among continuous-time subsystems with LES equilibria can be studied in an entirely analogous way to switched discrete systems. Let  $\mathcal{X}_p$  be an open subset of  $\mathbb{R}^n$  and assume that the vector field  $f_p : \mathcal{X}_p \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  of each subsystem is locally Lipschitz in its arguments. For each  $p \in \mathcal{P}$ , let  $x_p^* \in \mathcal{X}_p$  be a 0-input equilibrium point of (13) and assume that there exists a continuously differentiable function  $V_p : \mathcal{X}_p \rightarrow \mathbb{R}_+$  which, for all  $x \in \mathcal{X}_p$  satisfies (17) for suitable class- $\mathcal{K}_\infty$  functions  $\underline{\alpha}_p$  and  $\bar{\alpha}_p$ , and

$$\frac{\partial V_p}{\partial x} f_p(x, 0) \leq -\lambda_p V_p(x), \quad (47)$$

where  $\lambda_p > 0$ .

As in switched discrete systems, suppose that  $\bar{\kappa}_p > 0$  is such that (40) holds, and let  $\mathcal{X}$  defined by (41) be the domain within which we will work. Furthermore, we require the compositability assumption (42) to hold. With set constructions akin to those in Section IV-A and adopting (43) to define  $\mu$ , we can now present the main result of this section.

**Theorem 4.** *Consider the switched system (15) and assume that for each  $p \in \mathcal{P}$  there exists a continuously differentiable function  $V_p : \mathcal{X}_p \rightarrow \mathbb{R}_+$  which satisfies (17) for suitable class- $\mathcal{K}_\infty$  functions  $\underline{\alpha}_p$  and  $\bar{\alpha}_p$  and (47) for some  $\lambda_p > 0$ . Suppose further that there exist  $\kappa > 0$  and  $\bar{N}_0 \geq 1$  such that*

$$\mathcal{M}(\mu(\kappa)^{1+\bar{N}_0}\omega(\kappa)) \subset \mathcal{X}, \quad (48)$$

where  $\mu$  and  $\omega$  are given by (43) and (26), respectively,  $\mathcal{M}$  is the union over  $\mathcal{P}$  of the corresponding sublevel sets of  $V_p$ , and  $\mathcal{X}$  is the domain (41). Then, there exists  $\delta > 0$  such that for any disturbance signal  $d \in \mathcal{D}$  with  $\|d\|_\infty < \delta$ , and for any switching signal  $\sigma$  satisfying (16) with

$$\bar{N}_0 \geq N_0 \geq 1 \text{ and } N_a \geq \bar{N}_a ,$$

where  $\bar{N}_0$  satisfies (48) with the selected  $\kappa > 0$  and  $\bar{N}_a$  is defined by (37), the switched system (15) is practically stable with respect to the compact sets  $\Omega_1$  and  $\Omega_2$  defined by

$$\Omega_1 := \bigcap_{p \in \mathcal{P}} \mathcal{M}_p(\omega(\kappa)) \text{ and } \Omega_2 := \bigcup_{p \in \mathcal{P}} \mathcal{M}_p(\bar{\omega}(\|d\|_\infty)) ,$$

where

$$\bar{\omega}(\|d\|_\infty) := \mu(\kappa)^{1+N_0} \omega(\kappa) + \tilde{\alpha}(\|d\|_\infty) , \quad (49)$$

for a suitable class- $\mathcal{K}_\infty$  function  $\tilde{\alpha}$ .

A proof of Theorem 4 is provided in Section V-D below. We only mention here that the discussion following Theorem 3 applies to Theorem 4 as well.

## V. PROOFS

This section proves Theorems 1-4 and their corollaries.

### A. Proof of Theorem 1 and Corollaries 1 and 2

The following lemma establishes an important estimate that will be used in the proof of Theorem 1.

**Lemma 1.** Consider (3). Let  $\underline{k} \in \mathbb{Z}_+$  be the initial time and  $\{k_n\}_{n=1}^\infty$ ,  $k_n \in \mathbb{Z}_+$ , be a strictly monotonically increasing sequence of switching instants with  $k_1 > \underline{k}$ . Given  $\kappa > 0$ , define  $\mu(\kappa)$  by (27)-(28) and let

$$N := \inf\{n \in \mathbb{Z}_+ \cup \{\infty\} \mid x_{k_n} \in \mathring{\mathcal{M}}_{\sigma(k_n-1)}(\kappa)\} , \quad (50)$$

be the index of the first switching instant  $k_N$  for which (29) cannot be used. Assume that for each  $p \in \mathcal{P}$ , the function  $V_p$  satisfies (5) and (33) for some  $\lambda \in (0, 1)$ . Choose  $\epsilon \in (\lambda, 1)$  and assume that the switching signal satisfies Definition 2 for any  $N_0 \geq 1$  and  $N_a \geq \bar{N}_a$  where  $\bar{N}_a$  is given by (34). Then, for any  $x_{\underline{k}} \in \mathbb{R}^n$ ,  $d \in \mathcal{D}$ , and  $\underline{k} \leq k < k_N$ ,

$$V_{\sigma(k)}(x_k) \leq \mu^{N_0} \epsilon^{k-\underline{k}} V_{\sigma(\underline{k})}(x_{\underline{k}}) + \frac{\mu^{N_0}}{1-\epsilon} \hat{\alpha}(\|d\|_\infty) , \quad (51)$$

where  $\hat{\alpha}$  is the class- $\mathcal{K}_\infty$  function in (33). In addition, for  $\underline{k} \leq k < k_N$  the solutions of (3) satisfy

$$\|x_k - x_{\sigma(k)}^*\| \leq \beta(\|x_{\underline{k}} - x_{\sigma(\underline{k})}^*\|, k - \underline{k}) + \alpha(\|d\|_\infty) \quad (52)$$

for some  $\beta \in \mathcal{KL}$  and  $\alpha \in \mathcal{K}_\infty$ .

The proof of the lemma can be found in Appendix B. Now we are ready to present the proof of Theorem 1.

*Proof of Theorem 1.* The arguments of the proof refer to the set constructions of Section III-A. To simplify notation, the dependence on  $\kappa$  of  $\omega(\kappa)$  in (26) and of  $\mu(\kappa)$  in (28) will be dropped. Consider any (fixed) switching signal  $\sigma : \mathbb{Z}_+ \rightarrow \mathcal{P}$  satisfying Definition 2 for any  $N_0 \geq 1$  and  $N_a \geq \bar{N}_a$  where  $\bar{N}_a$  is given by (34). Without loss of generality, assume that

the system starts at  $k = 0$  and let  $\{k_1, k_2, \dots\}$  be the switching times. We first prove part (ii) and then part (i) of the theorem.

For part (ii), we distinguish the following cases:

*Case (a):*  $V_{\sigma(0)}(x_0) \leq \omega$ .

If  $N$  in (50) is unbounded, (29) can be used at all switching times and Lemma 1 ensures that (51) holds for all  $k \geq \underline{k} = 0$ . Since  $V_{\sigma(0)}(x_0) \leq \omega$  and  $\epsilon \in (\lambda, 1)$ , (51) implies

$$V_{\sigma(k)}(x_k) \leq \mu^{N_0} \omega + \frac{\mu^{N_0}}{1-\epsilon} \hat{\alpha}(\|d\|_\infty) =: \bar{\omega} \quad (53)$$

for all  $k \geq 0$ , showing that  $x_k \in \mathcal{M}(\bar{\omega})$  for all  $k \geq 0$  with  $\bar{\omega}$  as in (53). When, on the other hand,  $N$  is a finite number in  $\mathbb{Z}_+$ , Lemma 1 ensures that the estimate (53) holds over the interval  $[0, k_N]$ . By (50) it is clear that  $x_{k_N} \in \mathring{\mathcal{M}}_{\sigma(k_N-1)}(\kappa) \subset \mathcal{M}(\kappa)$ , which by Remark 1 implies

$$V_{\sigma(k_N)}(x_{k_N}) \leq \omega . \quad (54)$$

Since  $\mu \geq 1$ , the definition of  $\bar{\omega}$  by (53) implies that  $\omega \leq \bar{\omega}$ , so that by (54) we have  $V_{\sigma(k_N)}(x_{k_N}) \leq \bar{\omega}$ . As a result, when  $N$  is finite, the validity of the estimate (53) can be extended over the interval  $[0, k_N]$ . Now, considering  $k = k_N$  as the initial instant, the initial condition  $x_{k_N}$  satisfies (54) and the requirement for Case (a) holds at  $k = k_N$ . Hence, applying (51) of Lemma 1 with  $\underline{k} = k_N$  and propagating the same arguments as above from  $k_N$  onwards shows that  $V_{\sigma(k)}(x_k) \leq \bar{\omega}$  for all  $k \geq 0$ , proving that  $x_k$  never escapes from  $\mathcal{M}(\bar{\omega})$ . By the expression (53) for  $\bar{\omega}$ , choosing  $c := \mu^{N_0} \omega$  and  $\tilde{\alpha}(\|d\|_\infty) := (\mu^{N_0}/(1-\epsilon)) \hat{\alpha}(\|d\|_\infty)$  in (11) proves part (ii) for Case (a).

*Case (b)*  $V_{\sigma(0)}(x_0) > \omega$ .

As in Case (a), we distinguish between two subcases based on  $N$  defined by (50). When  $N$  is unbounded, Lemma 1 ensures that (51) holds for all  $k \geq \underline{k} = 0$ ; that is,

$$V_{\sigma(k)}(x_k) \leq \mu^{N_0} \epsilon^k V_{\sigma(0)}(x_0) + \frac{\mu^{N_0}}{1-\epsilon} \hat{\alpha}(\|d\|_\infty) . \quad (55)$$

If  $K \in \mathbb{Z}_+$  is such that

$$K \geq \frac{\ln(V_{\sigma(0)}(x_0)/\omega)}{\ln(1/\epsilon)} , \quad (56)$$

then  $\epsilon^k V_{\sigma(0)}(x_0) \leq \omega$  for all  $k \geq K$ , and (55) implies that the bound (53) holds for all  $k \geq K$ , establishing that  $x_k \in \mathcal{M}(\bar{\omega})$  for all  $k \geq K$ . If, on the other hand,  $N$  is a finite integer in  $\mathbb{Z}_+$ , then by the definition of  $N$  in (50) we have  $x_{k_N} \in \mathring{\mathcal{M}}_{\sigma(k_N-1)}(\kappa) \subset \mathcal{M}(\kappa)$ . By Remark 1, this condition implies that  $V_{\sigma(k_N)}(x_{k_N}) \leq \omega$  and the state  $x_{k_N}$  satisfies the conditions for Case (a). Hence, repeating the arguments of Case (a) from  $k_N$  onwards with  $x_{k_N}$  as the initial condition shows that  $x_k \in \mathcal{M}(\bar{\omega})$  for all  $k \geq k_N$  and with  $\bar{\omega}$  as defined in (53). Thus, choosing  $K = k_N$  proves part (ii) for Case (b) with the same choice for  $c$  and  $\tilde{\alpha}$  in (11) as in Case (a).

For part (i), when the initial condition  $x_0$  satisfies Case (a), then  $K = 0$  and the statement is vacuously true. If, on the other hand,  $x_0$  satisfies the conditions of Case (b), observe from the arguments above that  $K < k_N + 1$ . Indeed, if  $N$  is unbounded,  $k_N \rightarrow \infty$  and  $K$  is given by (56) while if  $N$  is a finite integer,  $K$  was selected equal to  $k_N$ . Hence, (52) in Lemma 1 holds for all  $k$  with  $\underline{k} = 0 \leq k < K$ , and the proof of part (i) is completed by choosing  $\beta, \alpha$  as in Lemma 1.  $\square$



**Remark 2.** It is of interest to discuss the behavior of the set  $\mathcal{M}(\bar{\omega})$  in the absence of disturbances; that is, when  $d_k = 0$  for all  $k \in \mathbb{Z}_+$ . In this case,  $\bar{\omega} = c = \mu^{N_0}\omega \geq \omega$ . It is clear from the proof of Theorem 1 that if the initial conditions  $x_0$  satisfy  $V_{\sigma(0)}(x_0) \leq \omega \leq c$ , the solution never leaves the set  $\mathcal{M}(c)$ . However, this does not imply that  $\mathcal{M}(c)$  is a forward invariant set of the 0-input system. Indeed, if the initial conditions  $x_0$  belong in the set  $\mathcal{M}(c)$  but satisfy  $\omega < V_{\sigma(0)}(x_0) \leq c$ , the solution may exit  $\mathcal{M}(c)$  before it returns to it forever, as the proof of Case (b) indicates. Example 2 in [50, Section V-B] illustrates that such behavior is possible.

*Proof of Corollary 1.* With the additional assumption<sup>7</sup> (12), (27) is bounded over the entire  $\mathbb{R}^n$  without the exclusion of an open set containing 0. Thus,  $\mu$  can be used for switches occurring at any  $x \in \mathbb{R}^n$  and hence  $k_N \rightarrow \infty$  in Lemma 1 so that (52) holds for all  $k \geq 0$ .  $\square$

*Proof of Corollary 2.* The proof is immediate by noting that  $K = 0$  for any  $x_0 \in \bigcap_{p \in \mathcal{P}} \mathcal{M}_p(\omega) =: \Omega_1$  due to the fact that Case (a) in the proof of Theorem 1 holds for this set. Further Remark 1 ensures that  $x_p^* \in \overset{\circ}{\Omega}_1$  for each  $p \in \mathcal{P}$ .  $\square$

### B. Proof of Theorem 2 and Corollaries 3 and 4

We begin with the following lemma, which is analogous to Lemma 1 and will be used in the proof of Theorem 2.

**Lemma 2.** Consider (15). Let  $\underline{t} \in \mathbb{R}_+$  be the initial time and  $\{t_n\}_{n=1}^\infty$ ,  $t_n \in \mathbb{R}_+$ , be a strictly monotonically increasing sequence of switching instants with  $t_1 > \underline{t}$ . Let  $x(t)$  be the solution of (15) for the corresponding switching signal. Given  $\kappa > 0$ , define  $\mu(\kappa)$  by (27)-(28) and let<sup>8</sup>

$$N := \inf\{n \in \mathbb{Z}_+ \cup \{\infty\} \mid x(t_n) \in \overset{\circ}{\mathcal{M}}_{\sigma(t_n^-)}(\kappa)\}, \quad (57)$$

be the index of the first switching instant  $t_N$  for which (29) cannot be used. Assume that for each  $p \in \mathcal{P}$ , the function  $V_p$  satisfies (17) and (36) for some  $\lambda > 0$ . Choose  $\epsilon \in (0, \lambda)$  and assume that the switching signal satisfies Definition 5 for any  $N_0 \geq 1$  and  $N_a \geq \bar{N}_a$  where  $\bar{N}_a$  is given by (37). Then, for any  $x(\underline{t}) \in \mathbb{R}^n$ ,  $d \in \mathcal{D}$ , and  $\underline{t} \leq t < t_N$ ,

$$V_{\sigma(t)}(x(t)) \leq \mu^{1+N_0} e^{-\epsilon(t-\underline{t})} V_{\sigma(\underline{t})}(x(\underline{t})) + \frac{\mu^{1+N_0}}{\epsilon} \hat{\alpha}(\|d\|_\infty), \quad (58)$$

where  $\hat{\alpha}$  is the class- $\mathcal{K}_\infty$  function in (36). In addition, for  $\underline{t} \leq t < t_N$  the solutions of (15) satisfy

$$\|x(t) - x_{\sigma(t)}^*\| \leq \beta(\|x(\underline{t}) - x_{\sigma(\underline{t})}^*\|, t - \underline{t}) + \alpha(\|d\|_\infty) \quad (59)$$

for some  $\beta \in \mathcal{KL}$  and  $\alpha \in \mathcal{K}_\infty$ .

The proofs of Lemma 2, Theorem 2, and Corollaries 3 and 4 follow analogously to the proofs of Lemma 1, Theorem 1, and Corollaries 1 and 2, respectively. Hence the details are omitted here; see [50] for complete proofs.

<sup>7</sup>Essentially, (12) ensures that  $V_p(x)$  does not converge to 0 substantially faster than  $V_q(x)$  as  $x \rightarrow 0$ .

<sup>8</sup>Define  $\sigma(t_n^-) := \lim_{t \nearrow t_n} \sigma(t)$ .

### C. Proof of Theorem 3

We begin with the following lemma regarding a geometric property of sublevel sets of the functions  $V_p$ ,  $p \in \mathcal{P}$  that will be used to prove Theorem 3.

**Lemma 3.** Consider the switched system (3) and assume that for each  $p \in \mathcal{P}$  there exists a locally Lipschitz function  $V_p : \mathcal{X}_p \rightarrow \mathbb{R}_+$  which satisfies (5) for suitable class- $\mathcal{K}_\infty$  functions  $\underline{\alpha}_p$  and  $\bar{\alpha}_p$ . Suppose further that there exist  $\kappa > 0$  and  $\bar{N}_0 \geq 1$  such that condition (44) is satisfied for the corresponding  $\mu$  and  $\omega$  as defined by (43) and (26), respectively. Then, for any  $\tilde{\alpha} \in \mathcal{K}_\infty$  and  $N_0 \in [1, \bar{N}_0]$ , there exists a  $\delta > 0$  such that

$$\mathcal{M}(\bar{\omega}(\delta)) \subset \mathcal{X}, \quad (60)$$

where  $\mathcal{X}$  is the domain defined by (41) and

$$\bar{\omega}(\delta) := \mu(\kappa)^{N_0} \omega(\kappa) + \tilde{\alpha}(\delta). \quad (61)$$

The proof for Lemma 3 can be found in Appendix B. We are now ready to present the proof of Theorem 3.

*Proof of Theorem 3.* We first show that there exists a  $\delta > 0$  such that for any disturbance  $d \in \mathcal{D}$  with  $\|d\|_\infty < \delta$  and any  $x \in \mathcal{M}_p(\bar{\kappa}_p)$ , the local Lyapunov function  $V_p : \mathcal{X}_p \rightarrow \mathbb{R}_+$  in the statement of Theorem 3 satisfies an ISS estimate

$$V_p(f_p(x, d)) \leq \lambda_p V_p(x) + \alpha_p(\|d\|_\infty), \quad (62)$$

for a suitable class- $\mathcal{K}_\infty$  function  $\alpha_p$ . This follows from [33, Theorem 2], which shows that a Lyapunov function  $V_p$  is also an ISS-Lyapunov function over a set where the map  $g_p := V_p \circ f_p : \mathcal{X}_p \times \mathbb{R}^m \rightarrow \mathbb{R}_+$  is (uniformly) Lipschitz. We claim: *Claim 1:* There exists a  $\delta > 0$  such that  $g_p$  is (uniformly) Lipschitz for all  $x \in \mathcal{M}_p(\bar{\kappa}_p)$  and  $d \in B_\delta(0)$ .

The proof of Claim 1 follows by noting that by Heine-Borel theorem,  $\mathcal{M}_p(\bar{\kappa}_p) \times \bar{B}_\delta(0) \subset \mathbb{R}^n \times \mathbb{R}^m$  is compact, and hence the locally Lipschitz function  $g_p$  is (uniformly) Lipschitz over this set. Using Claim 1 and repeating the steps in the proof of [33, Theorem 2] results in (62).

To proceed with the proof of Theorem 3, let  $1 \leq N_0 \leq \bar{N}_0$ . Further, choose  $\delta > 0$  as above and let  $x_0 \in \Omega_1$  with  $\Omega_1 = \bigcap_{p \in \mathcal{P}} \mathcal{M}_p(\omega(\kappa))$ . Then, using the estimate (62) and the average dwell-time conditions (45), follow steps analogous to those in the proofs of Lemma 1 and Theorem 1 Case (a), to define  $\bar{\omega}$  as in the proof of Theorem 1 Case (a) with a suitable  $\tilde{\alpha}$ . Use this  $\tilde{\alpha}$  and  $N_0$  in Lemma 3, shrinking  $\delta$  if necessary to satisfy (60), so that for any  $d \in \mathcal{D}$  with  $\|d\|_\infty < \delta$ , we have  $\mathcal{M}(\bar{\omega}(\|d\|_\infty)) \subset \mathcal{M}(\bar{\omega}(\delta)) \subset \mathcal{X}$ . Then, the arguments of Theorem 1 Case (a) imply that  $x_k \in \Omega_2$  for all  $k \in \mathbb{Z}_+$ , proving practical stability of (3) with respect to  $(\Omega_1, \Omega_2)$ .  $\square$

### D. Proof of Theorem 4

In what follows, we present Lemma 4, which is analogous to Lemma 3 and will be used to prove Theorem 4.

**Lemma 4.** Consider the switched system (15) and assume that for each  $p \in \mathcal{P}$  there exists a continuously differentiable function  $V_p : \mathcal{X}_p \rightarrow \mathbb{R}_+$  which satisfies (17) for suitable class- $\mathcal{K}_\infty$  functions  $\underline{\alpha}_p$  and  $\bar{\alpha}_p$ . Suppose further that there exist  $\kappa > 0$  and  $\bar{N}_0 \geq 1$  such that condition (48) is satisfied

for the corresponding  $\mu$  and  $\omega$  as defined by (43) and (26), respectively. Then, for any  $\tilde{\alpha} \in \mathcal{K}_\infty$  and  $N_0 \in [1, \bar{N}_0]$ , there exists a  $\delta > 0$  such that

$$\mathcal{M}(\bar{\omega}(\delta)) \subset \mathcal{X} ,$$

where  $\mathcal{X}$  is the domain defined by (41) and

$$\bar{\omega}(\delta) := \mu(\kappa)^{1+N_0}\omega(\kappa) + \tilde{\alpha}(\delta) .$$

The proof of Lemma 4 is identical to Lemma 3 and will be omitted. Next we present the proof of Theorem 4.

*Proof of Theorem 4.* As in the proof of Theorem 3, we begin with establishing an ISS estimate for the Lyapunov functions  $V_p$  involved in Theorem 4. In particular, there exists a  $\delta > 0$  such that for any  $d \in \mathcal{D}$  with  $\|d\|_\infty < \delta$  and any  $x \in \mathcal{M}_p(\bar{\kappa}_p)$ ,

$$\frac{\partial V_p}{\partial x} f_p(x, d) \leq -\lambda_p V_p(x) + \alpha_p(\|d\|_\infty) , \quad (63)$$

for a suitable class- $\mathcal{K}_\infty$  function  $\alpha_p \in \mathcal{K}_\infty$ . To show this estimate, first note that since  $\partial V_p / \partial x$  is continuous and  $\mathcal{M}_p(\bar{\kappa}_p)$  is compact, there is  $M_p > 0$  such that for all  $x \in \mathcal{M}_p(\bar{\kappa}_p)$ ,  $\|\partial V_p(x) / \partial x\| \leq M_p$ . Next, let  $\delta > 0$ , then  $\mathcal{M}_p(\bar{\kappa}_p) \times \bar{B}_\delta(0) \subset \mathbb{R}^n \times \mathbb{R}^m$  is compact. As locally Lipschitz functions on compact sets are (uniformly) Lipschitz, there exists a  $L_p > 0$  such that for all  $(x_1, d_1)$  and  $(x_2, d_2)$  in  $\mathcal{M}_p(\bar{\kappa}_p) \times \bar{B}_\delta(0)$ ,

$$\|f_p(x_1, d_1) - f_p(x_2, d_2)\| \leq L_p(\|x_1 - x_2\| + \|d_1 - d_2\|_\infty) .$$

Using these bounds and (47), we have

$$\begin{aligned} \frac{\partial V_p}{\partial x} f_p(x, d) &= \frac{\partial V_p}{\partial x} f_p(x, 0) + \frac{\partial V_p}{\partial x} (f_p(x, d) - f_p(x, 0)) \\ &\leq -\lambda_p V_p(x) + M_p L_p \|d\|_\infty , \end{aligned}$$

and choosing  $\alpha_p(\|d\|_\infty) = M_p L_p \|d\|_\infty$  results in (63).

The rest of the proof of Theorem 4 follows that of Theorem 3, but now in place of (62), Lemma 3, and Theorem 1, we use (63), Lemma 4, and Theorem 2, respectively.  $\square$

## VI. APPLICATION: ADAPTIVE LOCOMOTION OF LIMIT-CYCLE BIPEDAL WALKING ROBOTS

This section focuses on applying the switched system results developed above to realize practically stable gait adaptation in a 3D biped by switching among dynamic movement primitives, each corresponding to a limit-cycle locomotion behavior; see also [29] for more details.

### A. Overview: Adaptation by Switching

In the approach followed here, adaptation is achieved by switching among motion primitives that correspond to 0-input LES equilibria  $x_p^*$  of discrete (1) or continuous (13) dynamical systems together with the vector fields  $f_p$  that capture the corresponding dynamic behavior; that is,

$$\mathcal{G}_p := \{f_p, x_p^*\} , \quad p \in \mathcal{P} . \quad (64)$$

Converse Lyapunov theory [53], [54] then guarantees the existence of suitable Lyapunov functions, which are assumed to satisfy the composability condition (42); see Fig. 3(a) for a conceptual illustration. Then, the collection  $\mathbb{G} := \{\mathcal{G}_p \mid p \in \mathcal{P}\}$  of (64) constitutes a library of admissible motion primitives, which can be composed according to a higher-level supervisor to ensure adaptation. This process can be naturally modeled as a switched system with multiple equilibria (3) or (15), where the switching signal  $\sigma$  assigns to each time instant the motion primitive that needs to be executed.

Quantifying safety for such switched systems, can be challenging. Switching effectively causes the system to “shift” to a new equilibrium, and thus persistent switching in response to constantly varying environmental or task conditions causes the system to be in a “permanent” transient phase, never converging to any of the underlying equilibrium states. Theorems 3 and 4 resolve this problem by providing conditions that guarantee *practical stability* as in Definitions 3 and 6; this ensures safe operation in the sense that the system’s evolution is trapped within an explicitly characterized compact subset of the state space, even in the presence of disturbances. Furthermore, these conditions are stated as a pair of constants

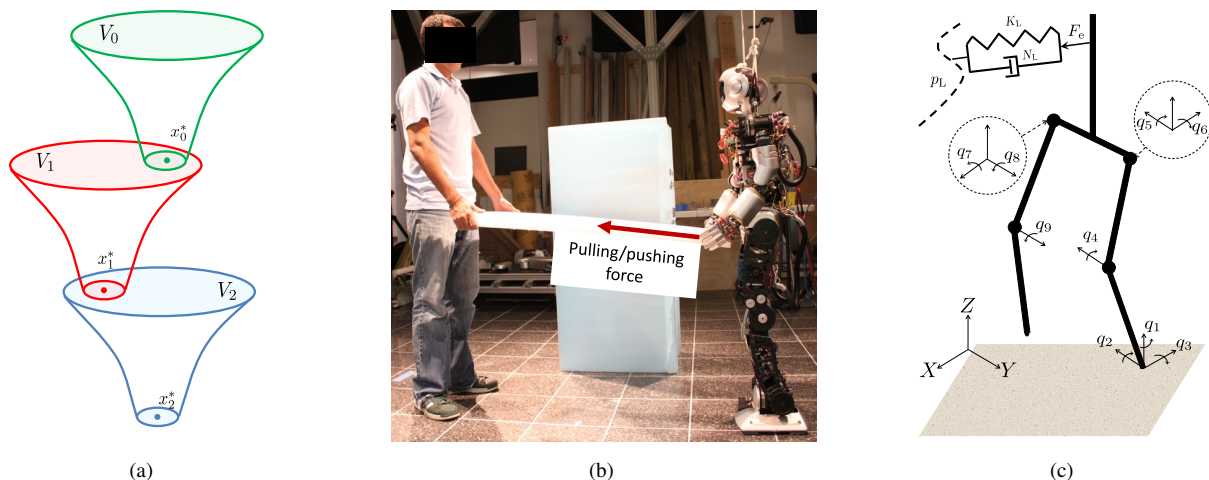


Fig. 3. (a) Illustration of dynamical motion primitives characterized by dynamical systems with LES equilibrium points, and their Lyapunov functions represented as funnels. (b) Collaborative object transportation between a human and a robot (photo courtesy of Prof. Paul Oh, University of Nevada, Las Vegas, NV). (c) Robot model with a choice of generalized coordinates when supported on left leg along with the impedance model for force generation.

$(\bar{N}_0, \bar{N}_a)$ , which, by (4) and (16), bound the rate of switching, effectively capturing the stability constraints associated with the low-level dynamics in a succinct form readily available to the high-level supervisor that decides when to switch.

Switched systems with multiple equilibria emerge in a wide range of applications. In what follows, we consider adaptive dynamic locomotion of bipedal robots as a concrete example of how the theoretical tools provided in the previous sections can be applied in practice. Dynamic locomotion behaviors can be mathematically modeled via limit cycles [55]—i.e., isolated periodic solutions in the system’s state space [56]—that can form motion primitives, the composition of which result in more complex movements for addressing the challenges of real-world tasks [14], [15]. Limit-cycle motion primitives can be naturally incorporated in the aforementioned setting, as they can be associated with point attractors of suitably constructed discrete-time systems via the method of Poincaré [56].

### B. Dynamic Locomotion for Collaboration

Our motivation stems from a class of tasks in which a bipedal robot physically collaborates with a leading human (or robotic) co-worker to transport an object over some distance; see Fig. 3(b). To safely accomplish such tasks, the robot needs to adapt its locomotion pattern according to the externally applied force supplied by the leader. We assume that the leader’s intention can be represented as a sufficiently smooth trajectory  $p_L : \mathbb{R}_+ \rightarrow \mathbb{R}^3$ , which is *not* explicitly available to the biped. Instead, the biped experiences an interaction force  $F_e : \mathbb{R}_+ \rightarrow \mathbb{R}^3$  that encodes important information about desired velocity and direction. As is common in the relevant literature [57], the interaction force  $F_e(t)$  is modeled by

$$F_e(t) = K_L(p_L(t) - p_E(t)) + N_L(\dot{p}_L(t) - \dot{p}_E(t)) , \quad (65)$$

where  $p_E(t)$  is the point at which the force is applied and  $K_L$  and  $N_L$  are suitable stiffness and damping matrices, respectively; see [28], [58], [59] for more details and Fig. 3(c).

In what follows, the exposition of modeling and control aspects is terse, but see [59], [60] for relevant details. The 3D bipedal robot model of Fig. 3(c) comprises nine degrees of freedom (DOFs), and its configuration can be described by  $q := (q_1, q_2, \dots, q_9)$ , as shown in Fig. 3(c). All DOFs except yaw  $q_1$  and pitch  $q_2$  are assumed to be actuated. Defining  $\hat{x} := (q, \dot{q}) \in \mathbb{R}^{18}$ , dynamic walking gaits can be represented as periodic solutions of the system with impulse effects

$$\Sigma : \begin{cases} \dot{\hat{x}} = f(\hat{x}) + g(\hat{x})u + g_e(\hat{x})F_e, & \text{if } \hat{x} \notin \mathcal{S} \\ \hat{x}^+ = \Delta(\hat{x}^-), & \text{if } \hat{x}^- \in \mathcal{S} \end{cases} , \quad (66)$$

where  $u$  are the inputs,  $(f, g, g_e)$  describe the swing phase dynamics under the influence of the external force  $F_e$ , and  $\mathcal{S}$  includes the states where the swing foot impacts the ground. The ensuing impact is captured by the map  $\Delta$  taking the states  $\hat{x}^-$  prior to impact to the states  $\hat{x}^+$  right after impact; see [60].

To design walking controllers, we adopt the hybrid zero dynamics (HZD) framework; see [59], [60] for details. In the absence of the external force ( $F_e \equiv 0$ ), a straightforward application of the HZD method can generate a LES limit cycle that corresponds to the biped walking along a straight line with

a desired nominal speed. Clearly, this (single) controller is not adequate for the task described above, and this is the case even when the intention of the leading collaborator exactly matches the biped’s gait speed and direction as Fig. 5(a) demonstrates. Indeed, since the leader’s intended trajectory  $p_L(t)$  can only be communicated to the biped via the interaction force (65), its application will inevitably cause a turning moment about the unactuated DOFs of the biped, eventually causing it to deviate from the intended trajectory as shown in Fig. 5(a). Accommodating the desired adaptability within a single control law can be challenging, particularly in the case where large deviations from the nominal conditions are required. However, the theoretical tools provided above can simplify the problem, and enhance adaptability by safely switching among different control laws in response to the externally applied force.

### C. Safe Adaptability: Switched Systems and Practical Stability

To adapt the biped’s motion to the interaction force, a family of feedback control laws  $\{\Gamma_p \mid p \in \mathcal{P}\}$  is designed based as in [59], [60], each resulting in a LES limit-cycle gait  $\mathcal{O}_p$ . Note that the application of Theorem 3 does not rely on the particular method chosen to stabilize the low-level locomotion behaviors; yet, the dimensional reduction afforded by the HZD method greatly simplifies computations. The end result is that each limit cycle  $\mathcal{O}_p$  can be associated with the 0-input fixed point of a reduced-order forced Poincaré map  $\rho_p$ —see [33] for a detailed definition—that gives rise to a discrete dynamical system evolving on  $\mathcal{S}_Z$  with dynamics

$$z_{k+1} = \rho_p(z_k, F_{e,k}) , \quad (67)$$

where  $k$  represents the stride number,  $F_{e,k} : \mathbb{R}_+ \rightarrow \mathbb{R}^3$  is the force<sup>9</sup> on the biped over the  $k$ -th stride,  $z$  are suitable coordinates for  $\mathcal{S}_Z$ , and  $z_p^*$  is the 0-input fixed point of  $\rho_p$  that corresponds to the limit cycle  $\mathcal{O}_p$ . In accordance with (64), the motion primitives in this example are  $\mathcal{R}_p = \{\rho_p, z_p^*\}$ , and switching among them as specified by a switching signal  $\sigma : \mathbb{Z}_+ \rightarrow \mathcal{P}$  gives rise to the switched discrete system with multiple equilibria

$$z_{k+1} = \rho_{\sigma(k)}(z_k, F_{e,k}) . \quad (68)$$

Note that, owing to the underlying geometry of the HZD method, the switched system (68) evolves in two dimensions.

1) *Application of Theorem 3 for Practical Stability:* For concreteness, we work with a library containing three motion primitives  $\mathcal{R}_p = \{\rho_p, z_p^*\}$  with  $p \in \mathcal{P} = \{0, 1, 2\}$  corresponding to the limit cycles  $\mathcal{O}_0$  for turning clockwise by  $30^\circ$ ,  $\mathcal{O}_1$  for walking straight, and  $\mathcal{O}_2$  for turning counterclockwise by  $30^\circ$ . Linearizing the 0-input system  $\rho_p(\cdot, 0)$  for each  $p \in \mathcal{P}$  about the corresponding fixed point  $z_p^*$  and using Lyapunov’s equation [46, equation (4.12)], results in a quadratic Lyapunov function  $V_p$  for the linearization. SoS programming is then used to verify that  $V_p$  is a Lyapunov function for the nonlinear system in a set containing the fixed

<sup>9</sup>The force is assumed to belong in the space of continuous and uniformly bounded functions  $\mathfrak{D} := \{F_e : \mathbb{R}_+ \rightarrow \mathbb{R}^3 \mid F_e \text{ is continuous, } \|F_e\|_\infty < \infty\}$ . Thus,  $\{F_{e,k}\}_{k \in \mathbb{Z}_+}$  is a sequence of functions in a Banach space  $\mathfrak{D}$  rather than the finite-dimensional Euclidean space  $\mathbb{R}^m$ ; see Appendix C for a rigorous discussion along the lines of [33].

points, so that the conditions (5), (39) and (42) required by Theorem 3 are satisfied for each  $p \in \mathcal{P}$ ; see [15] for details pertaining to SoS programming. The dashed ellipses of Fig. 4 show the corresponding sub-level sets  $\mathcal{M}_p(\bar{\kappa}_p)$  for each  $V_p$  for  $\bar{\kappa}_0 = 0.1736$ ,  $\bar{\kappa}_1 = 0.1415$ ,  $\bar{\kappa}_2 = 0.1120$ , respectively. Next, using the procedure explained in the second last paragraph of Section IV-A, we choose  $\kappa = 0.002$  and use Proposition 1 to compute upper bounds for  $\mu(\kappa)$  and  $\omega(\kappa)$ . With these bounds, we further choose  $\bar{N}_0 = 2$  so that the set  $\mathcal{M}(\mu(\kappa)\bar{N}_0\omega(\kappa))$  lies within  $\mathcal{X} := \mathring{\mathcal{M}}_0(\bar{\kappa}_0) \cap \mathring{\mathcal{M}}_1(\bar{\kappa}_1) \cap \mathring{\mathcal{M}}_2(\bar{\kappa}_2)$  as required by (44); see also Fig. 4. Finally, using (34) we compute  $\bar{N}_a = 0.99$ . Then, all the conditions of Theorem 3 are fulfilled, implying that for any initial condition in  $\Omega_1 = \mathcal{M}_0(\omega) \cap \mathcal{M}_1(\omega) \cap \mathcal{M}_2(\omega)$  and for switching signals that satisfy (4) with the chosen  $\bar{N}_0$  and  $\bar{N}_a$ , the evolution of the switched system (68) will never escape from the compact subset  $\Omega_2 = \mathcal{M}(\bar{\omega})$  of  $\mathcal{X}$ , provided that the external forces  $F_{e,k}$  are sufficiently small. Hence, the system is practically stable with respect to  $\Omega_1$  and  $\Omega_2$  under the influence of the externally applied force.

2) *Switching Policy and Adaptation:* As the leader's intended trajectory  $p_L(t)$ , is not directly available to the biped, the planner uses the external force as a cue for adaptation. Our switching policy estimates the ‘‘average’’ heading direction  $\Phi_k$  that the force  $F_e$  is pointing to over a stride, and then chooses the primitive that turns the biped towards this estimated heading. To compute  $\Phi_k$ , we integrate the force along the  $X$  and  $Y$  directions over a stride; see Fig. 3(c) for the global coordinate frame. Let  $t_0 = 0$  be the initial time and  $t_k$  be the time at the end of the  $k$ -th stride. Then, over the  $(k+1)$ -th stride, the integral of the force components are

$$F_k^X := \int_{t_k}^{t_{k+1}} F_e^X(t) dt, \quad F_k^Y := \int_{t_k}^{t_{k+1}} F_e^Y(t) dt,$$

which are used to compute the ‘‘average’’ heading as  $\Phi_k = \arctan(F_k^Y/F_k^X)$ . The switching policy is chosen to be  $\sigma(k+1) = \text{sign}(\Phi_k) + 1$  where the sign function returns -1, 0, 1 for negative, 0, and positive  $\Phi_k$ , respectively. We simulate the scenario shown in Fig. 5(b) where  $p_L(t)$  is represented by the red line, along which the leader intends to move at a constant speed of 0.65 m/s. Following the switching signal generated by

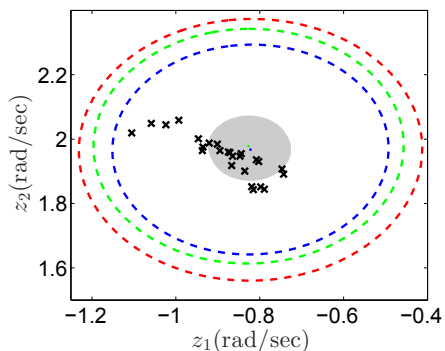


Fig. 4. Estimates of the BoA, i.e.,  $\mathcal{M}_p(\bar{\kappa}_p)$ , for the 0-input forced Poincaré maps  $\rho_p$ , and verification of (44). The BoA estimates  $\mathcal{M}_0(\bar{\kappa}_0)$ ,  $\mathcal{M}_1(\bar{\kappa}_1)$ , and  $\mathcal{M}_2(\bar{\kappa}_2)$  are the dashed red, green, and blue ellipses, respectively. The grey region is  $\mathcal{M}(\mu(\kappa)\bar{N}_0\omega(\kappa))$  in (44) for  $\kappa = 0.002$ ,  $\bar{N}_0 = 2$ . Black crosses are the solution of (68) for the simulation in Fig. 5(b).

our switching policy, the biped is able to adapt to the leader's intended trajectory in a safe manner, as verified by Fig. 4.

## VII. CONCLUSIONS

This paper proposed a framework for designing switching signals that ensure robustness under exogenous disturbances for switched continuous and discrete systems with multiple equilibria. It was shown that the solutions of such systems remain bounded if each subsystem is ISS and the switching signal satisfies an explicitly available average dwell-time constraint. Furthermore, relaxing the (global) ISS assumption to equilibria that are merely LES, it was proved that the resulting switched systems are practically stable provided again that the switching signal satisfies an explicit average dwell-time condition. Analytical computations of the bounds involved in the design of the switching signals can be facilitated in the case of quadratic Lyapunov functions. The theoretical results of this paper were implemented to realize safe gait adaptation of a 3D bipedal robot model in the presence of an external forcing signal. Although our motivation for studying this class of systems arises from robot motion planning via the composition of primitive movements, the results of this paper are relevant to a much broader class of applications which require switching among systems that do not share the same equilibrium point.

## APPENDIX A

*Proof of Proposition 1.* Since the functions  $V_p$  are quadratic, for all  $x \in \mathbb{R}^n$

$$\lambda_{\min}(S_p)\|x - x_p^*\|^2 \leq V_p(x) \leq \lambda_{\max}(S_p)\|x - x_p^*\|^2. \quad (69)$$

We will first show (31). From (25), (26) and since  $\mathcal{P}$  is a finite set, it follows that

$$\omega(\kappa) := \max_{p \in \mathcal{P}} \max_{x \in \mathcal{M}(\kappa)} V_p(x) = \max_{p, q \in \mathcal{P}} \max_{x \in \mathcal{M}_q(\kappa)} V_p(x). \quad (70)$$

Consider  $\max_{x \in \mathcal{M}_q(\kappa)} V_p(x)$ . For any  $x \in \mathcal{M}_q(\kappa)$ , we have

$$V_p(x) \leq \lambda_{\max}(S_p)\|x - x_p^*\|^2 \quad (71)$$

$$\leq \lambda_{\max}(S_p)(\|x - x_q^*\| + \|x_q^* - x_p^*\|)^2 \quad (72)$$

$$\leq \lambda_{\max}(S_p) \left( \sqrt{\frac{\kappa}{\lambda_{\min}(S_q)}} + \|x_q^* - x_p^*\| \right)^2, \quad (73)$$

where (71) follows from the second inequality of (69), which further leads to (72) by the use of triangle inequality. Finally, (73) follows from noting that for any  $x \in \mathcal{M}_q(\kappa)$ , the first inequality of (69) provides the bound  $\|x - x_q^*\| \leq \sqrt{\kappa/\lambda_{\min}(S_q)}$ , which, on using in (72), gives (73). As (73) holds for any  $x \in \mathcal{M}_q(\kappa)$ , we have shown that  $\max_{x \in \mathcal{M}_q(\kappa)} V_p(x)$  satisfies the bound in (73), which by (70) gives (31).

To show (32), from (27), (28) and the finite  $\mathcal{P}$  we have

$$\mu(\kappa) = \max_{p, q \in \mathcal{P}} \sup_{x \notin \mathring{\mathcal{M}}_p(\kappa)} \frac{V_q(x)}{V_p(x)}. \quad (74)$$

Consider  $\sup_{x \notin \mathring{\mathcal{M}}_p(\kappa)} V_q(x)/V_p(x)$ . For any  $x \notin \mathring{\mathcal{M}}_p(\kappa)$ ,

$$\frac{V_q(x)}{V_p(x)} \leq \frac{\lambda_{\max}(S_q)\|x - x_q^*\|^2}{\lambda_{\min}(S_p)\|x - x_p^*\|^2} \quad (75)$$

$$\leq \frac{\lambda_{\max}(S_q)}{\lambda_{\min}(S_p)} \left( 1 + \frac{\|x_p^* - x_q^*\|}{\|x - x_p^*\|} \right)^2, \quad (76)$$

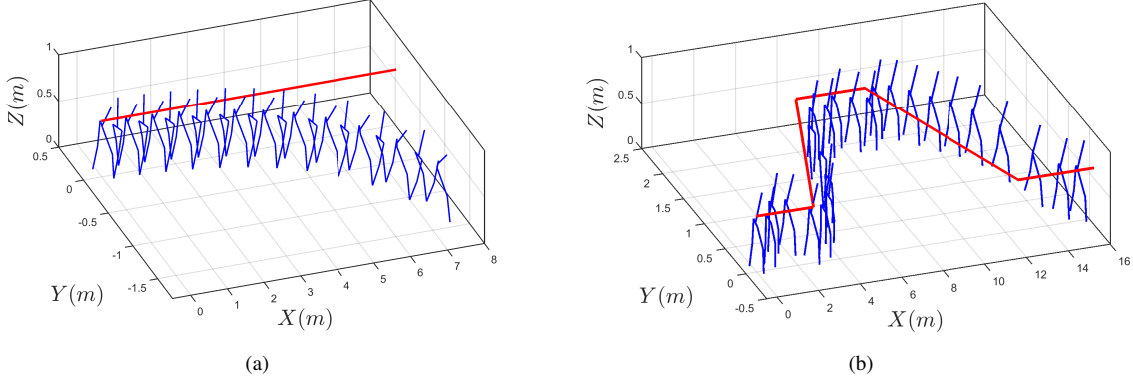


Fig. 5. Biped collaborating with a leader. The leader's intended trajectory  $p_L(t)$  is depicted by the red line and the blue stick figures represent the biped. (a) Single controller for walking straight. (b) Switching among limit cycles for walking straight, turning clockwise by  $30^\circ$ , and turning counterclockwise by  $30^\circ$  to adapt to the leader's intention.

where (75) follows from (69), and (76) follows from the triangle inequality. For  $x \notin \mathcal{M}_p(\kappa)$ ,  $V_p(x) \geq \kappa$  which by the second inequality of (69) gives  $\|x - x_p^*\| \geq \sqrt{\kappa/\lambda_{\max}(S_p)}$ . Using this in (76) followed by (74) gives (32).  $\square$

#### APPENDIX B PROOF OF LEMMAS

*Proof of Lemma 1.* The statement of Lemma 1 holds for an arbitrary initial time  $\underline{k}$ ; to avoid cumbersome expressions, we prove the result for  $\underline{k} = 0$  noting that the same proof carries to the case of an arbitrary  $\underline{k}$  by replacing  $k$  with  $k - \underline{k}$  in the expressions. We consider switching signals  $\sigma : \mathbb{Z}_+ \rightarrow \mathcal{P}$  that satisfy Definition 2 for  $N_0 \geq 1$  and  $N_a \geq \bar{N}_a$ , where  $\bar{N}_a$  is given by (34). Let  $\{k_1, k_2, \dots\}$  be a sequence of switching times for such signal. For notational compactness, define

$$G_a^b(r) := \begin{cases} \sum_{j=0}^{b-a-1} r^j = \frac{1-r^{b-a}}{1-r} & \text{if } b > a \\ 0 & \text{if } b = a \end{cases}.$$

where  $a, b \in \mathbb{Z}_+$ ,  $b \geq a$ , and  $0 < r < 1$ . Further, we denote  $N_\sigma(k, 0)$  by  $N_\sigma$  unless a different time window is specified.

Using (33) over the interval  $0 \leq k < k_1$  until the first switching occurs, results in

$$V_{\sigma(k)}(x_k) \leq \lambda^k V_{\sigma(0)}(x_0) + G_0^k(\lambda) \hat{\alpha}(\|d\|_\infty). \quad (77)$$

Now, since  $\mu \geq 1$  by (30) and  $\lambda < \epsilon$ , (77) results in

$$V_{\sigma(k)}(x_k) \leq \mu^{N_0} \epsilon^k V_{\sigma(0)}(x_0) + \frac{\mu^{N_0}}{1-\epsilon} \hat{\alpha}(\|d\|_\infty), \quad (78)$$

where we have used  $G_0^k(\lambda) \leq G_0^k(\epsilon) \leq \frac{1}{1-\epsilon}$ . Hence, (51) holds for all  $0 \leq k < k_1$ , completing the proof if  $N = 1$ .

Next, if  $N \neq 1$  so that  $x_{k_1} \notin \mathcal{M}_{\sigma(k_1-1)}$ , we can apply (29) to relate the values at the switching state  $x_{k_1}$  of the Lyapunov functions of the presently active system  $\sigma(k_1)$  and of the formerly active system  $\sigma(k_1 - 1) = \sigma(0)$ . Hence, using (77) first to obtain the bound  $V_{\sigma(k_1-1)}(x_{k_1}) \leq \lambda^{k_1} V_{\sigma(0)}(x_0) + G_0^{k_1}(\lambda) \hat{\alpha}(\|d\|_\infty)$ , we can then apply (29) to obtain  $V_{\sigma(k_1)}(x_{k_1}) \leq \mu \lambda^{k_1} V_{\sigma(0)}(x_0) + \mu G_0^{k_1}(\lambda) \hat{\alpha}(\|d\|_\infty)$ . This is used in (33) to write the following bound for  $k_1 \leq k < k_2$ ,

$$V_{\sigma(k)}(x_k) \leq \mu \lambda^k V_{\sigma(0)}(x_0) + \left( G_{k_1}^k(\lambda) + \mu \lambda^{k-k_1} G_0^{k_1}(\lambda) \right) \hat{\alpha}(\|d\|_\infty).$$

Inductively repeating this process for  $N_\sigma$  switches with  $1 \leq N_\sigma < N$  we have the following bound for  $k_{N_\sigma} \leq k < k_{N_\sigma+1}$ ,

$$V_{\sigma(k)}(x_k) \leq \mu^{N_\sigma} \lambda^k V_{\sigma(0)}(x_0) + \left( G_{k_{N_\sigma}}^k(\lambda) + \sum_{j=0}^{N_\sigma-1} \mu^{N_\sigma-j} \lambda^{k-k_{j+1}} G_{k_j}^{k_{j+1}}(\lambda) \right) \hat{\alpha}(\|d\|_\infty), \quad (79)$$

where  $k_j = 0$  for  $j = 0$ . We treat the state- and disturbance-dependent terms in the upper bound of (79) separately. For the state-dependent term, recall that  $\mu \geq 1$  by (30) and use (4) followed by  $N_a \geq \bar{N}_a$  where  $\bar{N}_a$  satisfies (34) to get,

$$\begin{aligned} \mu^{N_\sigma} \lambda^k V_{\sigma(0)}(x_0) &\leq \mu^{N_0} (\lambda \mu^{1/N_a})^k V_{\sigma(0)}(x_0) \\ &\leq \mu^{N_0} \epsilon^k V_{\sigma(0)}(x_0). \end{aligned} \quad (80)$$

To proceed with the disturbance-dependent term, first note that  $N_\sigma - j = N_\sigma(k, k_{j+1})$ . Hence, using (4) on  $N_\sigma(k, k_{j+1})$  followed by  $N_a \geq \bar{N}_a$  with  $\bar{N}_a$  given by (34) results in

$$\begin{aligned} \mu^{N_\sigma-j} &\leq \mu^{N_0} \mu^{(k-k_{j+1}) \ln(\epsilon/\lambda) / \ln(\mu)} \\ &\leq \mu^{N_0} (\epsilon/\lambda)^{k-k_{j+1}}. \end{aligned} \quad (81)$$

Using (81) in the summation in (79) gives

$$\begin{aligned} \sum_{j=0}^{N_\sigma-1} \lambda^{k-k_{j+1}} \mu^{N_\sigma-j} G_{k_j}^{k_{j+1}}(\lambda) &\leq \mu^{N_0} \sum_{j=0}^{N_\sigma-1} \epsilon^{k-k_{j+1}} G_{k_j}^{k_{j+1}}(\lambda) \\ &\leq \mu^{N_0} \sum_{j=0}^{N_\sigma-1} \epsilon^{k-k_{j+1}} G_{k_j}^{k_{j+1}}(\epsilon) \end{aligned} \quad (82)$$

where the last inequality follows from the fact that  $\lambda < \epsilon$ , hence  $G_{k_j}^{k_{j+1}}(\lambda) \leq G_{k_j}^{k_{j+1}}(\epsilon)$  with equality holding in the case when  $k_{j+1} = k_j + 1$ . It can be easily verified that

$$\epsilon^{k-k_{j+1}} G_{k_j}^{k_{j+1}}(\epsilon) = \epsilon^{k-k_{j+1}} + \epsilon^{k-k_{j+1}+1} + \dots + \epsilon^{k-k_j-1},$$

which, on summing from  $j = 0$  to  $j = N_\sigma - 1$  and after some algebraic manipulation, results in

$$\begin{aligned} \sum_{j=0}^{N_\sigma-1} \epsilon^{k-k_{j+1}} G_{k_j}^{k_{j+1}}(\epsilon) &= \sum_{j=k-k_1}^{k-1} \epsilon^j + \sum_{j=k-k_2}^{k-k_1-1} \epsilon^j + \dots \\ &\quad + \sum_{j=k-N_\sigma}^{k-k_{N_\sigma-1}-1} \epsilon^j = \sum_{j=k-k_{N_\sigma}}^{k-1} \epsilon^j. \end{aligned} \quad (83)$$

Using (83) in (82) gives

$$\sum_{j=0}^{N_\sigma-1} \lambda^{k-k_{j+1}} \mu^{N_\sigma-j} G_{k_j}^{k_{j+1}}(\lambda) \leq \mu^{N_0} \sum_{j=k-k_{N_\sigma}}^{k-1} \epsilon^j. \quad (84)$$

Additionally, as  $\mu \geq 1$  by (30) and  $\lambda < \epsilon$ ,

$$G_{k_{N_\sigma}}^k(\lambda) \leq \mu^{N_0} G_{k_{N_\sigma}}^k(\epsilon) = \mu^{N_0} \sum_{j=0}^{k-k_{N_\sigma}-1} \epsilon^j. \quad (85)$$

Thus, using (84) and (85) on the disturbance dependent term of the upper bound in (79) gives

$$\begin{aligned} & \left( G_{k_{N_\sigma}}^k(\lambda) + \sum_{j=0}^{N_\sigma-1} \lambda^{k-k_{j+1}} \mu^{N_\sigma-j} G_{k_j}^{k_{j+1}}(\lambda) \right) \hat{\alpha}(\|d\|_\infty) \\ & \leq \mu^{N_0} \sum_{j=0}^{k-1} \epsilon^j \hat{\alpha}(\|d\|_\infty) \leq \frac{\mu^{N_0}}{1-\epsilon} \hat{\alpha}(\|d\|_\infty). \end{aligned} \quad (86)$$

Hence, upper bounding (79) with (80) and (86) gives (51) for  $k_{N_\sigma} \leq k < k_{N_\sigma+1}$  for any  $1 \leq N_\sigma < N$ , i.e., for all  $k_1 \leq k < k_N$ . Further, by (78), (51) holds for  $0 \leq k < k_1$ . Hence, (51) holds for all  $0 \leq k < k_N$ .

Now we turn our attention to (52). Using (5) in (51),

$$\begin{aligned} \underline{\alpha}_{\sigma(k)}(\|x_k - x_{\sigma(k)}^*\|) & \leq \mu^{N_0} \epsilon^k \bar{\alpha}_{\sigma(0)}(\|x_0 - x_{\sigma(0)}^*\|) \\ & \quad + \frac{\mu^{N_0}}{1-\epsilon} \hat{\alpha}(\|d\|_\infty). \end{aligned} \quad (87)$$

As  $\underline{\alpha}_{\sigma(k)}^{-1} \in \mathcal{K}_\infty$  is monotonically increasing, by (87)

$$\begin{aligned} & \|x_k - x_{\sigma(k)}^*\| \\ & \leq \underline{\alpha}_{\sigma(k)}^{-1} \left( \mu^{N_0} \epsilon^k \bar{\alpha}_{\sigma(0)}(\|x_0 - x_{\sigma(0)}^*\|) + \frac{\mu^{N_0}}{1-\epsilon} \hat{\alpha}(\|d\|_\infty) \right) \\ & \leq \underline{\alpha}_{\sigma(k)}^{-1} \left( 2\mu^{N_0} \epsilon^k \bar{\alpha}_{\sigma(0)}(\|x_0 - x_{\sigma(0)}^*\|) \right) + \underline{\alpha}_{\sigma(k)}^{-1} \left( \frac{2\mu^{N_0}}{1-\epsilon} \hat{\alpha}(\|d\|_\infty) \right) \end{aligned} \quad (88)$$

where the last inequality follows by [61, Lemma 14] with  $\epsilon = 1$ . Observe that the first term in (88) is in class  $\mathcal{KL}$  while the second is in class  $\mathcal{K}_\infty$ . Let  $s \in \mathbb{R}_+$  and define  $\beta \in \mathcal{KL}$  as

$$\beta(s, k) := \max_{p, q \in \mathcal{P}} \underline{\alpha}_p^{-1} \left( 2\mu^{N_0} \epsilon^k \bar{\alpha}_q(s) \right). \quad (89)$$

Further define  $\alpha \in \mathcal{K}_\infty$  as

$$\alpha(s) := \max_{p \in \mathcal{P}} \underline{\alpha}_p^{-1} \left( \frac{2\mu^{N_0}}{1-\epsilon} \hat{\alpha}(s) \right). \quad (90)$$

Using (89) and (90) in (88) gives (52).  $\square$

*Proof of Lemma 3.* For the sake of notational convenience, let

$$\hat{\mathcal{X}} := \bigcup_{p \in \mathcal{P}} \mathcal{M}_p(\bar{\kappa}_p) \setminus \bigcap_{p \in \mathcal{P}} \mathring{\mathcal{M}}_p(\bar{\kappa}_p),$$

which, by the definition of  $\mathcal{X}$  in (41), condition (44), and  $N_0 \leq \bar{N}_0$ , does not contain any  $x \in \mathcal{M}(\bar{\omega}(0))$ ; see Fig. 2 where  $\hat{\mathcal{X}}$  is illustrated by the light yellow region. Therefore,

$$\forall p \in \mathcal{P}, \quad V_p(x) > \bar{\omega}(0), \quad \forall x \in \hat{\mathcal{X}}. \quad (91)$$

Let  $\bar{\kappa}$  be defined as

$$\bar{\kappa} := \min_{p \in \mathcal{P}} \min_{x \in \hat{\mathcal{X}}} V_p(x), \quad (92)$$

which exists because  $\hat{\mathcal{X}}$  is compact and  $\mathcal{P}$  is finite. Then,

$$\forall p \in \mathcal{P}, \quad \mathcal{M}_p(\bar{\kappa}) \subseteq \mathcal{M}_p(\bar{\kappa}_p), \quad (93)$$

which follows from  $\bar{\kappa} \leq \min_{p \in \mathcal{P}} \bar{\kappa}_p$ , which can be justified by a contradiction argument<sup>10</sup>.

Now, from (91), it follows that  $\bar{\kappa} > \bar{\omega}(0)$ . Shrink  $\delta$  if necessary to ensure that  $0 < \delta < \bar{\alpha}^{-1}(\bar{\kappa} - \bar{\omega}(0))$ , where  $\bar{\alpha} \in \mathcal{K}_\infty$  is as in (61). Then, for any  $p \in \mathcal{P}$ , and  $x \in \mathcal{M}_p(\bar{\omega}(\delta))$ ,

$$V_p(x) \leq \bar{\omega}(0) + \bar{\alpha}(\delta) < \bar{\kappa}. \quad (94)$$

Hence,

$$\forall p \in \mathcal{P}, \quad \mathcal{M}_p(\bar{\omega}(\delta)) \subset \mathcal{M}_p(\bar{\kappa}). \quad (95)$$

Furthermore, using (95) in (93) gives,

$$\forall p \in \mathcal{P}, \quad \mathcal{M}_p(\bar{\omega}(\delta)) \subset \mathcal{M}_p(\bar{\kappa}_p). \quad (96)$$

To complete the proof, we use a contradiction to show that (60) holds with the choice of  $\delta$  as above. Assume, *ad absurdum*, that this is not true. Then, there must exist a  $\hat{x} \in \mathcal{M}(\bar{\omega}(\delta))$  which lies in the complement of  $\mathcal{X} = \bigcap_{p \in \mathcal{P}} \mathring{\mathcal{M}}_p(\bar{\kappa}_p)$ . Since  $\hat{x} \in \mathcal{M}(\bar{\omega}(\delta))$  we have that  $\hat{x} \in \mathcal{M}_p(\bar{\omega}(\delta))$  for some  $p \in \mathcal{P}$ , which by (96) gives  $\hat{x} \in \mathcal{M}_p(\bar{\kappa}_p)$  for that  $p$ . This combined with  $\hat{x} \notin \bigcap_{p \in \mathcal{P}} \mathring{\mathcal{M}}_p(\bar{\kappa}_p)$  imply that  $\hat{x} \in \hat{\mathcal{X}}$ . Hence, since  $\hat{x} \in \hat{\mathcal{X}}$ , by (92) we have  $V_p(\hat{x}) \geq \bar{\kappa}$ , and since  $\hat{x} \in \mathcal{M}_p(\bar{\omega}(\delta))$ , by (94) we have  $V_p(\hat{x}) < \bar{\kappa}$ , arriving at a contradiction and completing the proof.  $\square$

## APPENDIX C

### DISCRETE DISTURBANCE SIGNAL IN A BANACH SPACE

Even though the results developed for discrete switched systems in Section II-A and Section IV-A of this paper are for disturbance signals in a Euclidean space, they also apply to the more general case of disturbances in an arbitrary Banach space. Such signals often arise in the study of robustness of periodic phenomena [33], as it did in the robotic application studied in Section VI.

Let  $(\mathfrak{D}, \|\cdot\|_{\mathfrak{D}})$  be a Banach space [62] and consider a discrete disturbance signal  $d: \mathbb{Z}_+ \rightarrow \mathfrak{D}$ , which belongs to the set of *bounded* disturbances  $\mathcal{D} := \{d: \mathbb{Z}_+ \rightarrow \mathfrak{D} \mid \|d\|_\infty := \sup_{k \in \mathbb{Z}_+} \|d_k\|_{\mathfrak{D}} < \infty\}$ . For this class of disturbances, the proofs of all the theorems presented in this paper follow identically as before except for Claim 1 in the proof of Theorem 3. Hence, in what follows we prove Claim 1 for the case of disturbances in an arbitrary (possibly infinite-dimensional) Banach space.

*Proof of Claim 1.* As  $V_p$  and  $f_p$  are locally Lipschitz in their arguments, their composition  $g_p := V_p \circ f_p$  is locally Lipschitz as well. Hence, for any  $(x, 0) \in \mathcal{M}_p(\bar{\kappa}_p) \times \mathfrak{D}$ , there exists a  $\delta_x > 0$  and  $L_x > 0$  such that  $\|g_p(x_1, d_1) - g_p(x_2, d_2)\| \leq$

<sup>10</sup>Suppose, *ad absurdum*, that  $\bar{\kappa} > \min_{p \in \mathcal{P}} \bar{\kappa}_p$ . Without loss of generality, let  $\bar{\kappa}_1 = \min_{p \in \mathcal{P}} \bar{\kappa}_p$ . Then, by the definition (92) of  $\bar{\kappa}$ , for each  $x \in \hat{\mathcal{X}}$ ,  $V_1(x) \geq \bar{\kappa} > \min_{p \in \mathcal{P}} \bar{\kappa}_p = \bar{\kappa}_1$ , implying that every point in  $\hat{\mathcal{X}}$  is strictly outside  $\mathcal{M}_1(\bar{\kappa}_1)$ ; i.e.,  $\hat{\mathcal{X}} \cap \mathcal{M}_1(\bar{\kappa}_1) = \emptyset$ . On the other hand,  $\hat{\mathcal{X}}$  includes all the boundary sets  $\partial \mathcal{M}_p(\bar{\kappa}_p)$ , because, by definition,  $\partial \mathcal{M}_p(\bar{\kappa}_p)$  cannot be in  $\mathring{\mathcal{M}}_p(\bar{\kappa}_p)$ . But since  $\mathcal{M}_1(\bar{\kappa}_1)$  is closed, it must contain  $\partial \mathcal{M}_1(\bar{\kappa}_1)$  so that  $\hat{\mathcal{X}} \cap \mathcal{M}_1(\bar{\kappa}_1) \neq \emptyset$  leading to a contradiction with  $\hat{\mathcal{X}} \cap \mathcal{M}_1(\bar{\kappa}_1) = \emptyset$ .

$L_x \|(x_1 - x_2, d_1 - d_2)\|$ , for any<sup>11</sup>  $x_1, x_2 \in B_{\delta_x}(x)$  and  $d_1, d_2 \in B_{\delta_x}(0) \subset \mathfrak{D}$ . Construct an open cover  $\bigcup_{x \in \mathcal{M}_p(\bar{\kappa}_p)} B_{\delta_x/2}(x)$  of  $\mathcal{M}_p(\bar{\kappa}_p)$  which is compact, hence there exists  $\hat{x}_1, \hat{x}_2, \dots, \hat{x}_N$  such that  $\mathcal{M}_p(\bar{\kappa}_p) \subset \bigcup_{i=1}^N B_{\delta_i/2}(\hat{x}_i)$  where  $\delta_i := \delta_{\hat{x}_i}$  and define  $\delta := \min\{\delta_1/2, \dots, \delta_N/2\}$ .

Consider  $x_1, x_2 \in \mathcal{M}_p(\bar{\kappa}_p)$  and  $d_1, d_2 \in B_\delta(0) \subset \mathfrak{D}$ . Then, the following two cases arise.

*Case (a):* There exists an  $i \in \{1, \dots, N\}$  such that  $x_1, x_2 \in B_{\delta_i}(\hat{x}_i)$ .

As  $d_1, d_2 \in B_\delta(0) \subset B_{\delta_i/2}(0)$ , and by the assumption of this case  $x_1, x_2 \in B_{\delta_i}(\hat{x}_i)$ , we can use the Lipschitz continuity of  $g_p$  in the  $\delta_i$  neighborhood of  $(\hat{x}_i, 0)$  to obtain  $\|g_p(x_1, d_1) - g_p(x_2, d_2)\| \leq L_i \|(x_1 - x_2, d_1 - d_2)\|$  where  $L_i := L_{\hat{x}_i}$ . Define  $\tilde{L} := \max\{L_1, \dots, L_N\}$ , then we can express the Lipschitz bound as

$$\|g_p(x_1, d_1) - g_p(x_2, d_2)\| \leq \tilde{L} \|(x_1 - x_2, d_1 - d_2)\|. \quad (97)$$

*Case (b):* There does not exist any  $i \in \{1, \dots, N\}$  such that  $x_1, x_2 \in B_{\delta_i}(\hat{x}_i)$ .

To obtain the Lipschitz bound in this case we first need to establish uniform boundedness of  $g_p$  over  $\mathcal{M}_p(\bar{\kappa}_p) \times B_\delta(0) \subset \mathcal{X}_p \times \mathfrak{D}$ . Note that  $g_p(\cdot, 0) : \mathcal{X}_p \rightarrow \mathcal{X}_p$  is Lipschitz on the compact set  $\mathcal{M}_p(\bar{\kappa}_p)$  as it is locally Lipschitz in its arguments. Hence, there exists a  $\tilde{L} > 0$  such that  $\|g_p(y_1, 0) - g_p(y_2, 0)\| \leq \tilde{L} \|y_1 - y_2\|$  for any  $y_1, y_2 \in \mathcal{M}_p(\bar{\kappa}_p)$ . Further, using the boundedness (compactness) of  $\mathcal{M}_p(\bar{\kappa}_p) \subset \mathbb{R}^n$ , there exists a  $r > 0$  such that  $\|y_1 - y_2\| \leq r$  for any  $y_1, y_2 \in \mathcal{M}_p(\bar{\kappa}_p)$ . As  $\mathcal{M}_p(\bar{\kappa}_p) \subset \bigcup_{i=1}^N B_{\delta_i/2}(\hat{x}_i)$ , there exist  $\hat{x}_m$  and  $\hat{x}_n$  such that  $\|x_1 - \hat{x}_n\| < \delta_n/2$  and  $\|x_2 - \hat{x}_m\| < \delta_m/2$ . Then,

$$\begin{aligned} & \|g_p(x_1, d_1) - g_p(x_2, d_2)\| \\ &= \|g_p(x_1, d_1) - g_p(\hat{x}_n, 0) + g_p(\hat{x}_n, 0) - g_p(\hat{x}_m, 0) \\ &\quad + g_p(\hat{x}_m, 0) - g_p(x_2, d_2)\| \\ &\leq \|g_p(x_1, d_1) - g_p(\hat{x}_n, 0)\| + \|g_p(\hat{x}_n, 0) - g_p(\hat{x}_m, 0)\| \\ &\quad + \|g_p(\hat{x}_m, 0) - g_p(x_2, d_2)\| \\ &\leq L_n(\|x_1 - \hat{x}_n\| + \|d_1\|) + \tilde{L} \|\hat{x}_n - \hat{x}_m\| \\ &\quad + L_m(\|x_2 - \hat{x}_m\| + \|d_2\|) \\ &\leq 2\tilde{L}(r + \delta) + \tilde{L}r =: M. \end{aligned} \quad (98)$$

Also, it can be noted that  $\|x_1 - x_2\| \geq \delta$  which can be shown by the way of contradiction. Suppose  $\|x_1 - x_2\| < \delta$ . Let  $\hat{x}_n$  be such that  $\|x_1 - \hat{x}_n\| < \delta_n/2$  which exists because  $\mathcal{M}_p(\bar{\kappa}_p) \subset \bigcup_{i=1}^N B_{\delta_i/2}(\hat{x}_i)$ . Then, adding and subtracting this  $\hat{x}_n$  in  $\|x_1 - x_2\|$ , and using reverse triangle inequality gives

$$\|x_2 - \hat{x}_n\| - \|x_1 - \hat{x}_n\| \leq \|x_1 - \hat{x}_n + \hat{x}_n - x_2\| < \delta$$

which leads to  $\|x_2 - \hat{x}_n\| < \delta + \|x_1 - \hat{x}_n\| < \delta_n/2 + \delta_n/2 = \delta_n$  implying that  $x_2 \in B_{\delta_n}(\hat{x}_n)$ , which along with the fact that  $x_1 \in B_{\delta_n}(\hat{x}_n)$  leads to a contradiction with the assumption

<sup>11</sup>Notation: We use  $B_\delta(a)$  to denote an open-ball of radius  $\delta$  centered at  $a$ . This notation can be used for open-balls in  $\mathbb{R}^n$ , as well as  $\mathfrak{D}$ . It will be clear from context the space to which the ball belongs.

of Case (b). Hence,  $\|x_1 - x_2\| \geq \delta$  which is used in (98) to obtain

$$\begin{aligned} & \|g_p(x_1, d_1) - g_p(x_2, d_2)\| \\ &\leq M \leq \frac{M}{\delta} \|x_1 - x_2\| \leq \frac{M}{\delta} \|(x_1 - x_2, d_1 - d_2)\|. \end{aligned} \quad (99)$$

With the bounds (97) and (99) in Case (a) and (b), respectively, let  $L := \max\{\tilde{L}, M/\delta\}$  to obtain

$$\|g_p(x_1, d_1) - g_p(x_2, d_2)\| \leq L \|(x_1 - x_2, d_1 - d_2)\|,$$

for any  $x_1, x_2 \in \mathcal{M}_p(\bar{\kappa}_p)$  and  $d_1, d_2 \in B_\delta(0)$ .  $\square$

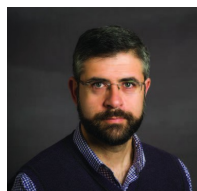
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