Practical Stability of Switched Systems With Multiple Equilibria Under Disturbances

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Abstract— **Our objective in this paper is to establish robustness to disturbances for continuous-time switched systems with multiple equilibria (SSME) while being unaware of the disturbances. We provide an average dwell-time bound which can be computed *without* explicit knowledge of the disturbance. Switching signals that satisfy this bound ensure safe operation of the switched system for sufficiently small disturbances by the notion of practical stability. In essence, this paper establishes the robustness property that safe evolution of the SSME in the absence of disturbances results in the safety of the SSME under mild disturbances. Our motivation for studying SSMEs under disturbances arises from robotics, where certain motion planning problems require switching among controllers under unknown or unmodeled disturbances. However, these results are applicable to a much broader class of applications.**

I. INTRODUCTION

A switched system comprises a family of dynamical subsystems where the choice of the active subsystem is governed by a switching signal. Such systems arise in a wide variety of applications where switching among various modes of operation is required—such as electronics [1], automotive control [2], robotics [3], and air-traffic control [4], among many others. In this paper, we will study the robustness of such switched systems to disturbance signals.

Owing to the versatility of switched systems as a modeling tool for a variety of practical applications, the study of their stability has attracted attention in the past few decades. However, most of the research efforts have been focused on the class of switched systems in which the individual subsystems share a common equilibrium point; see [5], [6] for such systems. Various applications demand switching among subsystems that do not share the same equilibrium resulting in *switched systems with multiple equilibria* (SSME). Examples where these systems arise include, robot motion planning [7], [8], cooperative manipulation [9], power control of wireless networks [10], and neuron models [11].

When switching occurs among systems that do not share a common equilibrium point, the state cannot be expected to converge to any single point unless switching ceases. To ensure boundedness for solutions of SSMEs under persistent switching, a class of switching signals characterized by a dwell-time bound was identified in continuous- [12] and discrete- [8] time switched systems; dwell-time bounds have also been proposed in [13] and [14] to ensure practical stability of such systems. Switching among subsystems with multiple invariant sets, rather than equilibria, was studied in [15]. However, none of the aforementioned efforts considers switching in the presence of disturbances. To address this, the authors' previous work [16] established ultimate boundedness of solutions for SSMEs with input-to-state stable (ISS) subsystems. Nevertheless, [16] requires the knowledge of a global ISS-Lyapunov function for each subsystem, which may be difficult or impossible to obtain in practical applications. For example, robotic systems rarely exhibit global stability properties and disturbances are often unmodeled and/or unknown. Motivated by this example, in this paper we relax the requirement of global ISS-Lyapunov functions to local exponential Lyapunov functions and establish safety guarantees in the form of *practical stability*. This notion of stability captures the fact that even though various systems lack stable equilibrium behaviors, they still operate safely, and can be considered stable in a "practical" sense [17].

Our interest in perturbed SSME arises from their application to problems in robotic locomotion where a motion plan must be devised using dynamical motion primitives. Such primitives are characterized by point attractors or limit cycles [18], and their composition induces a SSME structure to the dynamics of the robot [7]–[9], [19]. The ability to safely switch among such primitives can enhance the capability of robots to navigate cluttered environments [7], [8], [20], adapt to external signals [21]–[23], traverse uneven terrain [24], [25], transition among various locomotion patterns [19], [26], [27], and many others. The results developed in this paper facilitate such robotic applications by providing rigorous guarantees of safe switching among dynamical motion primitives in the face of *unknown* disturbances.

In this paper, we study switching among subsystems with locally exponentially stable equilibria under disturbances. It is shown that the average dwell-time bound computed in the *absence* of disturbances will ensure "safe" operation of the switched system despite the presence of disturbances. In effect, this allows us to obtain the average dwell-time bound using Lyapunov functions, alleviating the need to find an ISS-Lyapunov function, as required by [16]. Our notion of safety requires trapping the solutions in a suitable compact set for all time, from where the state would recover the nominal (equilibiurm) operation of the active subsystem, should switching ceases and disturbances vanish. Using the notion of practical stability, we establish robust switching for SSME (Theorem 1) which naturally extends to switched systems that share a common equilibrium point (Corollary 1).

Notation: The set of reals and integers are denoted by $\mathbb{R}$ and $\mathbb{Z}$, and non-negative reals and integers are denoted by

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$\mathbb{R}_+$ and $\mathbb{Z}_+$, respectively. For $x \in \mathbb{R}^n$, the Euclidean norm is denoted by $\|x\|$. A Euclidean open-ball of radius $\delta > 0$ centered at x is denoted by $B_\delta(x) \subset \mathbb{R}^n$. Let $\mathcal{A} \subseteq \mathbb{R}^n$ be a set, then its interior is denoted by $\mathring{\mathcal{A}}$ while its closure is denoted by $\overline{\mathcal{A}}$. The disturbance signal is denoted as a mapping $d : \mathbb{R}_+ \to \mathbb{R}^m$ with norm $\|d\|_\infty := \sup_{t\in\mathbb{R}_+} \|d(t)\|$. It is assumed that d belongs to $\mathcal{D} := \{d : \mathbb{R}_+ \to \mathbb{R}^m \mid d \text{ is piecewise continuous}, \ \|d\|_\infty < \infty\}$. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}_\infty$ if it is continuous, strictly increasing, $\alpha(0) = 0$, and $\lim_{s\to\infty} \alpha(s) = \infty$.

II. Switched Systems with Multiple Equilibria

This section introduces the class of switched systems that we study and the requisite notions of safety and stability.

A. Family of Dynamical Systems

Consider a finite family of continuous-time dynamical systems indexed by a finite set $\mathcal{P}$,

$$\dot{x}(t) = f_p(x(t), d(t)) \ , \qquad p \in \mathcal{P} \ , \tag{1}$$

where $\mathcal{X}_p \subseteq \mathbb{R}^n$ is an open and connected subset of $\mathbb{R}^n$ and $x \in \mathcal{X}_p$ is the state, $f_p : \mathcal{X}_p \times \mathbb{R}^m \to \mathbb{R}^n$ is the vector field of the p-th system that is locally Lipschitz in its arguments, and $d \in \mathcal{D}$ is a disturbance signal; see the notation at the end of Section I. Each system exhibits a unique (in $\mathcal{X}_p$) equilibrium point x_p^* in the absence of disturbances, i.e. $f_p(x_p^*, 0) = 0$ for all $p \in \mathcal{P}$. The majority of the switched systems literature assumes that each member of the family shares a common equilibrium point [6]. Here, we relax this assumption and allow for the possibility that if $p \neq q$, then $x_p^* \neq x_q^*$. Furthermore, we assume that the equilibrium point of each system is exponentially stable in the absence of disturbances and is endowed with a continuously differentiable Lyapunov function $V_p : \mathcal{X}_p \to \mathbb{R}_+$, that satisfies for all $x \in \mathcal{X}_p$,

$$\underline{\alpha}(\|x - x_p^*\|) \leq V_p(x) \leq \overline{\alpha}(\|x - x_p^*\|) \ , \tag{2}$$

$$\frac{\partial V_p}{\partial x} f_p(x, 0) \leq -\lambda V_p(x) \ , \tag{3}$$

where $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$ and $\lambda > 0$. Note that the assumption of uniform (over $\mathcal{P}$) bounds and rate of convergence does not result in any loss of generality. Indeed, even if the bounds $\underline{\alpha}_p$, $\overline{\alpha}_p$, and the rate of convergence λ_p are different for each $p \in \mathcal{P}$, we can always choose $\underline{\alpha}(\cdot) := \min_{p\in\mathcal{P}} \underline{\alpha}_p(\cdot)$, $\overline{\alpha}(\cdot) := \max_{p\in\mathcal{P}} \overline{\alpha}_p(\cdot)$, and $\lambda := \min_{p\in\mathcal{P}} \lambda_p > 0$.

Associated with each subsystem is a compact inner approximation $\mathcal{B}_p$ of its basin-of-attraction (BoA). In many practical situations [8], [22], [26], such approximations can be obtained through semi-definite programming, such as sum-of-squares [28]. We can shrink $\mathcal{B}_p$ further, if necessary, to address system limitations—such as actuator saturation, friction bounds, joint limits and other constraints. For the sake of convenience, define

$$\underline{\mathcal{B}} := \bigcap_{p\in\mathcal{P}} \mathcal{B}_p, \tag{4}$$

as the intersection of all $\mathcal{B}_p$. Then, to ensure the feasibility of switching among the subsystems we assume that each equilibrium point x_p^* lies in $\mathring{\underline{\mathcal{B}}}$.

B. Switched System

Let $\sigma : \mathbb{R}_+ \to \mathcal{P}$ be a right-continuous switching signal which, at any given time instant $t \in \mathbb{R}_+$, executes the $\sigma(t) \in \mathcal{P}$ member of the family giving rise to a switched system,

$$\dot{x}(t) = f_{\sigma(t)}(x(t), d(t)) \ ; \tag{5}$$

see [16, Section II-B] for a discussion of solutions of (5), which are continuous in time. Our objective is to ensure "safe" operation of the switched system (5) despite disturbances. We consider the system to be safe if (i) its solution is trapped within a compact set for all time and (ii) it is always in a state from which it will converge back to the equilibrium of the most recent active system, should switching ceases and disturbances vanish. This would ensure that the system always operates in a regime from where the nominal behavior can be recovered. We make this intuitive notion of *safety* for (5) mathematically precise in the following definition.

Definition 1: A solution $x(t)$ of (5) is considered to be *safe* if $x(t) \in \mathcal{B}_{\sigma(t)}$ for all $t \geq 0$.

A sufficient (but not necessary) condition for Definition 1 to hold is that for all $t \geq 0$,

$$x(t) \in \bigcap_{p\in\mathcal{P}} \mathcal{B}_p =: \underline{\mathcal{B}} \ . \tag{6}$$

We will study the class of switching signals σ that ensure the safety of the switched system (5) by establishing (6). In particular, we will achieve this by identifying σ that lead to the *practical stability* [17] of (5) with appropriate sets. Next, we provide a definition of practical stability.

Definition 2: The switched system (5) is *practically stable* with respect to the sets Ω_1 and Ω_2 with $\Omega_1 \subset \Omega_2$, if $x(0) \in \Omega_1$ implies $x(t) \in \Omega_2$ for all $t \geq 0$.

We will characterize this class of switching signals by the notion of average dwell time, introduced in [29], which is formalized in the following definition.

Definition 3: A switching signal $\sigma(t)$ has *average dwell-time* $N_\mathrm{a} > 0$ if the number $N_\sigma(t, \underline{t}) \in \mathbb{Z}_+$ of switches over any interval $[\underline{t}, t) \subset \mathbb{R}_+$ satisfies

$$N_\sigma(t, \underline{t}) \leq N_0 + \frac{t - \underline{t}}{N_\mathrm{a}} \ , \qquad \forall t \geq \underline{t} \geq 0 \tag{7}$$

where $N_0 \geq 1$ is a finite constant.

Intuitively, the average dwell time N_a represents the average time-gap between any two consecutive switches. Hence, a large N_a leads to slower switching. The role of N_0 is to provide additional flexibility to switch faster than the average in some time intervals, which can be compensated by slower switching in others.

III. Set Constructions

In this section we introduce certain set constructions and constants which will be used throughout the paper. A detailed discussion of the set construction can be found in [16, Section III], hence, the exposition here will be terse.

Let V_p be a Lyapunov function for the p-th system in the family (1) which satisfies (2) and (3). For the sake of notational convenience, we define

$$\mathcal{M}_p(\kappa) := \{x \in \mathbb{R}^n \mid V_p(x) \leq \kappa\} \ , \tag{8}$$

as the κ-sublevel set of V_p, and their union over $\mathcal{P}$ as

$$\mathcal{M}(\kappa) := \bigcup_{p\in\mathcal{P}} \mathcal{M}_p(\kappa) \ . \tag{9}$$

Further, define

$$\omega(\kappa) := \max_{p\in\mathcal{P}} \max_{x\in\mathcal{M}(\kappa)} V_p(x) \ , \tag{10}$$

which by construction entails that $\mathcal{M}(\kappa) \subseteq \mathcal{M}_p(\omega(\kappa))$ for each $p \in \mathcal{P}$; see [16, Remark 1] for justification. An illustration of this set construction can be found in Fig. 1.

To quantify the ratio by which the value of the Lyapunov function changes on switching from subsystem p to any other subsystem in $\mathcal{P}$, define the positive constant

$$\mu_p(\kappa) := \max_{q\in\mathcal{P}} \max_{x\in\mathcal{B}_p\setminus\mathring{\mathcal{M}}_p(\kappa)} \frac{V_q(x)}{V_p(x)} \ , \tag{11}$$

which is well-defined as $\mathcal{P}$ is finite and $\mathcal{B}_p \setminus \mathring{\mathcal{M}}_p(\kappa)$ is compact. The max over x in (11) is restricted within $\mathcal{B}_p \subset \mathcal{X}_p$ because when switching-out from p—in accordance with Definition 1—we expect the state to be within $\mathcal{B}_p$. The exclusion of the open set $\mathring{\mathcal{M}}_p(\kappa)$ containing x_p^* from the max is to eliminate the possibility of the denominator approaching zero while the numerator is non-zero; note that $V_p(x_p^*) = 0$ while $V_q(x_p^*) > 0$. Further, we can define

$$\mu(\kappa) := \max_{p\in\mathcal{P}} \mu_p(\kappa) \ , \tag{12}$$

which is independent of p. The interchangeability of p and q implies that $\mu(\kappa) \geq 1$; see [16, Section III-A] for details.

IV. Main Results

With the constructions and definitions in Section III, we are now ready to state the main result of this paper.

Theorem 1: Consider the switched system (5). For each $p \in \mathcal{P}$ assume that there exists a continuously differentiable function V_p that satisfies (2) and (3). Further, assume that there exist $\overline{N}_0 \geq 1$ and $\kappa > 0$, such that

$$\mathcal{M}(\mu(\kappa)^{1+\overline{N}_0}\omega(\kappa)) \subset \mathring{\underline{\mathcal{B}}} \ , \tag{13}$$

where μ, ω, and $\underline{\mathcal{B}}$ are as in (12), (10), and (4), respectively. Then, there exists a $\delta > 0$ such that for any $d \in \mathcal{D}$ with $\|d\|_\infty < \delta$, and any switching signal σ that satisfies (7) with

$$1 \leq N_0 \leq \overline{N}_0 \quad \text{and} \quad N_{\mathrm{a}} \geq \overline{N}_{\mathrm{a}} := \frac{\ln \mu(\kappa)}{\lambda - \epsilon} \ , \tag{14}$$

where λ is as in (3) and $\epsilon \in (0, \lambda)$, the switched system (5) is practically stable with respect to the sets

$$\Omega_1 := \bigcap_{p\in\mathcal{P}} \mathcal{M}_p(\omega(\kappa)) \text{ and } \Omega_2(\|d\|_\infty) := \mathcal{M}(\bar{\omega}(\|d\|_\infty)), \tag{15}$$

where $\Omega_2(\|d\|_\infty) \subset \mathring{\underline{\mathcal{B}}}$ and

$$\bar{\omega}(\|d\|_\infty) := \mu(\kappa)^{1+\overline{N}_0}\omega(\kappa) + \alpha(\|d\|_\infty) \ , \tag{16}$$

for some $\alpha \in \mathcal{K}_\infty$.

The proof of Theorem 1 is provided in Section V. Let us now discuss some important aspects of this theorem. To

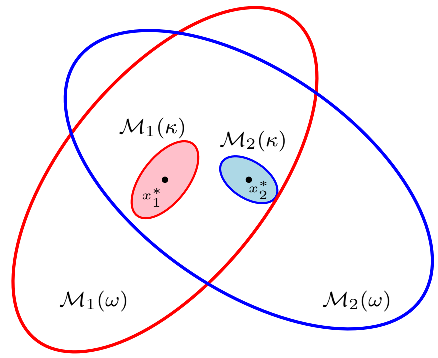


Fig. 1. Illustration of the set construction. The sublevel sets for system 1 are in red and the sublevel sets of system 2 are in blue.

verify that the hypothesis of Theorem 1 holds, we do not require the knowledge of the disturbance—besides the fact that $d \in \mathcal{D}$ and the vector fields f_p in (1) are locally Lipschitz with respect to d. The average dwell-time bound $\overline{N}_{\mathrm{a}}$ defined by (14) and the constant $\overline{N}_0$ in (13) are independent of the disturbance as well, lending a *disturbance-agnostic* nature to Theorem 1. Furthermore, from the expression of $\bar{\omega}$ in (16) it follows that the set $\mathcal{M}(\mu(\kappa)^{1+\overline{N}_0}\omega(\kappa))$ in (13) is the compact trapping set $\Omega_2(0)$ in (15) for $d \equiv 0$. Hence, we can interpret Theorem 1 as a robustness result which states that, if the 0-input ($d \equiv 0$) switched system is practically stable with respect to Ω_1 and $\Omega_2(0)$, then, for sufficiently small disturbances, the disturbed switched system (5) is also practically stable with respect to Ω_1 and $\Omega_2(\|d\|_\infty)$, as defined in (15).

To facilitate implementation of Theorem 1 in applications, we provide the procedure to obtain $\overline{N}_0$ and $\overline{N}_{\mathrm{a}}$. First, choose a $\kappa > 0$ and compute $\omega(\kappa)$ and $\mu(\kappa)$ as defined in (10) and (12), respectively. Then, obtain $\overline{N}_0$ for which (13) holds and compute $\overline{N}_{\mathrm{a}}$ using (14). In the event that (13) does not hold for any $\overline{N}_0 \geq 1$ given the selected $\kappa > 0$ and the resulting $\omega(\kappa)$ and $\mu(\kappa)$, try a different κ. It is worth noting that (10) and (12) cannot—in general—be written analytically, and their numerical computation can be challenging, particularly for high-dimensional systems. Nevertheless, if the Lyapunov functions are quadratic, then analytical upper bounds for $\omega(\kappa)$ and $\mu(\kappa)$ can be obtained by [16, Proposition 1].

Next, we provide a result analogous to Theorem 1 for switched systems with a common equilibrium point.

Corollary 1: Consider the switched system (5). Assume that for each $p \in \mathcal{P}$, $x_p^* = 0$ and that there exists a continuously differentiable function V_p that satisfies (2) and (3). Furthermore, we assume that for any $p, q \in \mathcal{P}$,

$$\limsup_{\|x\|\to 0} \frac{V_p(x)}{V_q(x)} < \infty \ . \tag{17}$$

Then, there exist $\kappa > 0$ and $\delta > 0$ such that for any $d \in \mathcal{D}$ with $\|d\|_\infty < \delta$, and any switching signal σ that satisfies (7) with $N_0 \geq 1$ and $N_{\mathrm{a}} \geq \overline{N}_{\mathrm{a}}$, where $\overline{N}_{\mathrm{a}}$ is as defined in (14), the switched system (5) is practically stable with respect to Ω_1 and $\Omega_2(\|d\|_\infty) \subset \mathring{\underline{\mathcal{B}}}$ as defined in (15).

The proof of Corollary 1 is provided in Section V. A few remarks are now in order. It can be noted that, similarly to Theorem 1, Corollary 1 does not require knowledge of the disturbance to provide safety guarantees in the sense of Definition 1. Another fact worth noting is that the existence of $\kappa > 0$ that satisfies (13) is not an assumption in Corollary 1; the proof of Corollary 1 establishes that, in the common equilibrium case, there always exists such a κ. A consequence of this fact is that $N_0 \geq 1$ can be chosen arbitrarily. Finally, we would like to point out that assumption (17) appears regularly in the literature of switched systems with a common equilibrium, albeit in a different form. Essentially, it ensures that, on switching, the Lyapunov functions remain bounded; see [6, p. 58].

V. Proof of the Main Results

In the forthcoming lemma we will show that the Lyapunov function V_p is also an ISS-Lyapunov function for all $x \in \mathcal{B}_p$ and sufficiently small disturbances.

Lemma 1: Let V_p be a continuously differentiable Lyapunov function that satisfies (2) and (3) for the 0-input system (1). Then, there exists a $\delta_p > 0$ such that for any $x \in \mathcal{B}_p$ and $d \in \mathcal{D}$ with $\|d\|_\infty < \delta_p$ the following holds

$$\frac{\partial V_p}{\partial x} f_p(x,d) \leq -\lambda V_p(x) + \tilde{\alpha}(\|d\|_\infty) \ , \tag{18}$$

where $\tilde{\alpha} \in \mathcal{K}_\infty$.

Proof: Since V_p is continuously differentiable and $\mathcal{B}_p$ is compact, there exists a $M > 0$ such that

$$\left\| \frac{\partial V_p}{\partial x}(x) \right\| \leq M \tag{19}$$

for all $x \in \mathcal{B}_p$. Next, let $\delta_p > 0$. Then, $\mathcal{B}_p \times \overline{B}_{\delta_p}(0) \subset \mathcal{X}_p \times \mathbb{R}^m$ is a compact set. Since, locally Lipschitz functions on compact sets are Lipschitz, it follows that there exists $L > 0$ such that for all (x_1, d_1) and (x_2, d_2) in $\mathcal{B}_p \times \overline{B}_{\delta_p}(0)$,

$$\|f_p(x_1,d_1) - f_p(x_2,d_2)\| \leq L\big(\|x_1 - x_2\| + \|d_1 - d_2\|_\infty\big) \ , \tag{20}$$

and we have

$$\begin{aligned} \frac{\partial V_p}{\partial x} f_p(x,d) &= \frac{\partial V_p}{\partial x} f_p(x,0) + \frac{\partial V_p}{\partial x}(f_p(x,d) - f_p(x,0)) \\ &\leq -\lambda V_p(x) + ML\|d\|_\infty \ , \end{aligned} \tag{21}$$

where (21) follows by using (3), (19), and (20). Finally, choosing $\tilde{\alpha}(\|d\|_\infty) = ML\|d\|_\infty$ completes the proof. ∎

Now we are ready to present the proof of Theorem 1.

Proof: [Theorem 1] This proof follows from suitably restricting [16, Corollary 4]. By Lemma 1, for each $p \in \mathcal{P}$, there exists a $\delta_p > 0$ such that for any $d \in \mathcal{D}$ with $\|d\|_\infty < \delta_p$, the functions V_p satisfy (2) and (18) for $x \in \mathcal{B}_p$. Restricting attention to the interior $\mathring{\underline{\mathcal{B}}}$ of the intersection $\underline{\mathcal{B}} := \bigcap_{p\in\mathcal{P}} \mathcal{B}_p$, which was defined in (4), and taking $\delta' = \min_{p\in\mathcal{P}} \delta_p$ we have that for any $d \in \mathcal{D}$ with $\|d\|_\infty < \delta'$, the functions V_p, $p \in \mathcal{P}$, satisfy the conditions (2) and (18) for all $x \in \mathring{\underline{\mathcal{B}}}$. These conditions are identical to those required by [16, Corollary 4], with the difference that here they hold only "locally" over $\mathring{\underline{\mathcal{B}}} \subset \mathbb{R}^n$ instead of "globally" over $\mathbb{R}^n$ as in [16, Corollary 4]. The rest of the proof is dedicated to identifying an appropriate bound $\delta > 0$ on the disturbance signal $d \in \mathcal{D}$ that allows the translation of [16, Corollary 4] to the local domain.

By the assumptions of the theorem, there exists a pair $(\overline{N}_0, \kappa)$ with $\overline{N}_0 \geq 1$ and $\kappa > 0$ such that condition (13) is satisfied; in what follows we work with such pair $(\overline{N}_0, \kappa)$. Define the time it takes a solution to exit the set $\mathring{\underline{\mathcal{B}}}$ as

$$T := \inf\{t \geq 0 \mid x(t) \in \mathbb{R}^n \backslash \mathring{\underline{\mathcal{B}}}\} \ . \tag{22}$$

Following steps identical to those in the proof of [16, Corollary 4], we have that for any switching signal that satisfies (7) with N_0 and N_a as in (14), and for any $d \in \mathcal{D}$ with $\|d\|_\infty < \delta'$ with $\delta' = \min_{p\in\mathcal{P}} \delta_p$,

$$x(0) \in \bigcap_{p\in\mathcal{P}} \mathcal{M}_p(\omega(\kappa)) \implies x(t) \in \mathcal{M}(\bar{\omega}(\|d\|_\infty)) \ , \tag{23}$$

for all $0 \leq t < T$, where $\bar{\omega}$ is defined by (16) for a suitable $\alpha \in \mathcal{K}_\infty$. We make the following claim.

Claim: Let $\delta \in (0, \delta')$ be chosen so that

$$\mathcal{M}(\bar{\omega}(\delta)) \subset \mathring{\underline{\mathcal{B}}} \ . \tag{24}$$

Then, for any solution of (5) for switching signals that satisfy (7) with N_0 and N_a as in (14), any $x(0) \in \bigcap_{p\in\mathcal{P}} \mathcal{M}_p(\omega(\kappa))$, and for any $d \in \mathcal{D}$ with $\|d\|_\infty < \delta$, we have that $T \to \infty$ and therefore (23) holds for all $t \geq 0$.

This claim implies that (5) is practically stable with respect to the sets Ω_1 and Ω_2 defined in (15), and its proof is provided in the appendix. To complete the proof of the theorem, we only need to satisfy the assumptions of the aforementioned claim; in particular, we will establish the existence of a $\delta > 0$ such that (24) holds.

Let $\partial\underline{\mathcal{B}}$ be the boundary of $\underline{\mathcal{B}}$. Condition (13) implies that $\mathcal{M}(\mu(\kappa)^{1+\overline{N}_0}\omega(\kappa))$ lies entirely in the interior of $\underline{\mathcal{B}}$, and so $\partial\underline{\mathcal{B}}$ does not contain any $x \in \mathcal{M}(\mu(\kappa)^{1+\overline{N}_0}\omega(\kappa))$. Hence,

$$\forall p \in \mathcal{P}, \quad V_p(x) > \mu(\kappa)^{1+\overline{N}_0}\omega(\kappa), \quad \forall x \in \partial\underline{\mathcal{B}} \ . \tag{25}$$

Let

$$\overline{\kappa} := \min_{p\in\mathcal{P}} \min_{x\in\partial\underline{\mathcal{B}}} V_p(x) \ , \tag{26}$$

which is well-defined because $\partial\underline{\mathcal{B}}$ is compact and $\mathcal{P}$ is finite. From (25), it follows that $\overline{\kappa} > \mu(\kappa)^{1+\overline{N}_0}\omega(\kappa)$. Let $0 < c < \overline{\kappa} - \mu(\kappa)^{1+\overline{N}_0}\omega(\kappa)$, and shrink $0 < \delta < \delta'$ if necessary to ensure $0 < \delta < \alpha^{-1}(c)$, where the function α is the one participating in the definition of $\bar{\omega}$ by (16). Then, for any $p \in \mathcal{P}$, and any $x \in \mathcal{M}_p(\bar{\omega}(\delta))$,

$$\begin{aligned} V_p(x) &\leq \mu(\kappa)^{1+\overline{N}_0}\omega(\kappa) + \alpha(\delta) \\ &< \mu(\kappa)^{1+\overline{N}_0}\omega(\kappa) + c < \overline{\kappa} \ . \end{aligned} \tag{27}$$

With the choice of δ as above we claim that (24) holds. Assume *ad absurdum* that (24) does not hold. Then, there exists a $\hat{x} \in \mathcal{M}_p(\bar{\omega}(\delta))$ for some $p \in \mathcal{P}$, which lies in $\partial\underline{\mathcal{B}}$. However, this implies that $V_p(\hat{x}) \geq \overline{\kappa}$ leading to a contradiction with (27). Hence, (24) holds for $0 < \delta < \min\{\delta', \alpha^{-1}(c)\}$ with α a suitable class-$\mathcal{K}_\infty$ function and $c \in (0, \overline{\kappa} - \mu(\kappa)^{1+\overline{N}_0}\omega(\kappa))$ for $\overline{\kappa}$ defined by (26), thus satisfying the requirement of the claim. Practical stability

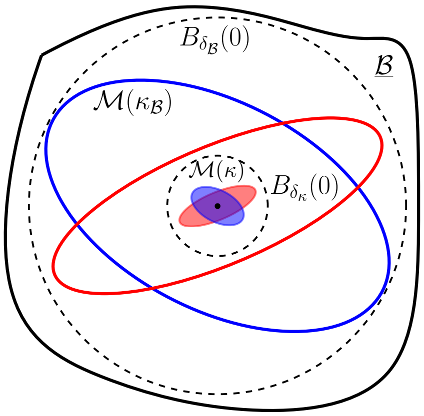


Fig. 2. Illustration of the set construction in the proof of Corollary 1. The sublevel sets for system 1 are in red and for system 2 are in blue.

of (5) with respect to the sets defined in (15) follows from that claim, completing the proof of Theorem 1. ■

Now we present the proof of Corollary 1.

Proof: [Corollary 1] We need to show that there exists a $\kappa > 0$ such that (13) holds; then, Corollary 1 follows directly from Theorem 1. Before finding such a κ, we provide a bound $\overline{\mu}$ on $\mu(\kappa)$ that is independent of κ. Note that $\mu_p(\kappa)$ in (11) monotonically increases as κ decreases. Hence,

$$\mu(\kappa) \leq \limsup_{\kappa \to 0} \mu(\kappa) =: \overline{\mu} < \infty \ , \tag{28}$$

where the boundedness is an outcome of (17).

Next, we proceed with finding a $\kappa > 0$ such that (13) holds. First, construct a ball of radius $\delta_{\mathcal{B}} > 0$, centered at 0 that lies within $\mathring{\underline{\mathcal{B}}}$; such an open ball exists because $0 \in \mathring{\underline{\mathcal{B}}}$ and $\mathring{\underline{\mathcal{B}}}$ is open. Then, we "fit" a union of sublevel sets $\mathcal{M}(\kappa_{\mathcal{B}})$ inside $B_{\delta_{\mathcal{B}}}(0)$ as shown by the larger (unfilled) ellipses in Fig. 2. To establish the existence of such a union of sublevel sets, let $\underline{\alpha} \in \mathcal{K}_\infty$ be as in (2), then choose $\kappa_{\mathcal{B}} \in (0, \underline{\alpha}(\delta_{\mathcal{B}}))$ which ensures that $\mathcal{M}(\kappa_{\mathcal{B}}) \subset B_{\delta_{\mathcal{B}}}(0)$. Now, we construct the smaller open ball in Fig. 2 by choosing $\delta_\kappa \in (0, \overline{\alpha}^{-1}(\kappa_{\mathcal{B}}/\overline{\mu}^{1+N_0}))$ where $\overline{\alpha} \in \mathcal{K}_\infty$ is defined in (2), $\overline{\mu}$ is given by (28) and $N_0 \geq 1$ is arbitrary. This ensures that $B_{\delta_\kappa}(0) \subset \bigcap_{p \in \mathcal{P}} \mathcal{M}_p(\kappa_{\mathcal{B}}/\overline{\mu}^{1+N_0})$. Finally, we construct $\mathcal{M}(\kappa)$ shown by the smaller (filled) ellipses in Fig. 2 by choosing $\kappa \in (0, \underline{\alpha}(\delta_\kappa))$, which ensures that $\mathcal{M}(\kappa) \subset B_{\delta_\kappa}(0) \subset \bigcap_{p \in \mathcal{P}} \mathcal{M}_p(\kappa_{\mathcal{B}}/\overline{\mu}^{1+N_0})$. Therefore, for any $p \in \mathcal{P}$ and $x \in \mathcal{M}(\kappa)$, we have $V_p(x) < \kappa_{\mathcal{B}}/\overline{\mu}^{1+N_0}$, which, by (10), ensures that $\omega(\kappa) < \kappa_{\mathcal{B}}/\overline{\mu}^{1+N_0}$, alternatively expressed as $\overline{\mu}^{1+N_0}\omega(\kappa) < \kappa_{\mathcal{B}}$. Hence, by (28), we also have $\mu(\kappa)^{1+N_0}\omega(\kappa) < \kappa_{\mathcal{B}}$ implying that $\mathcal{M}(\mu(\kappa)^{1+N_0}\omega(\kappa)) \subset \mathcal{M}(\kappa_{\mathcal{B}}) \subset \mathring{\underline{\mathcal{B}}}$, establishing (13) with $\overline{N}_0 = N_0 \geq 1$. ■

VI. Example

Consider the family of continuous-time systems indexed by $p \in \mathcal{P} := \{1, 2, 3\}$,

$$\dot{x}(t) = A_p x(t) + B_p + d(t) \ , \tag{29}$$

where $x \in \mathbb{R}^2$, $A_p \in \mathbb{R}^{2\times 2}$, $B_p \in \mathbb{R}^2$, and $d : \mathbb{R}_+ \to \mathbb{R}^2$ is a disturbance signal. The family of systems (29) is borrowed from [12, Section 4.2] and hence, in the interest of space, we do not provide the exact expressions of A_p and B_p here. We only mention that the equilibrium points of the members of (29) are $x_1^* = [0, 1]^{\mathrm{T}}$, $x_2^* = [-1, 0]^{\mathrm{T}}$, and $x_3^* = [1, 0]^{\mathrm{T}}$ and that each equilibrium is exponentially stable as A_p is Hurwitz for each $p \in \mathcal{P}$. Suppose now that for safe operation, we require solutions to stay within a compact set $\underline{\mathcal{B}}$ plotted by the dashed black ellipse in Fig. 3. In what follows, we compute an average dwell-time bound for the family of systems using Theorem 1, and demonstrate that the solutions stay within $\mathring{\underline{\mathcal{B}}}$ in the presence of sufficiently small disturbances.

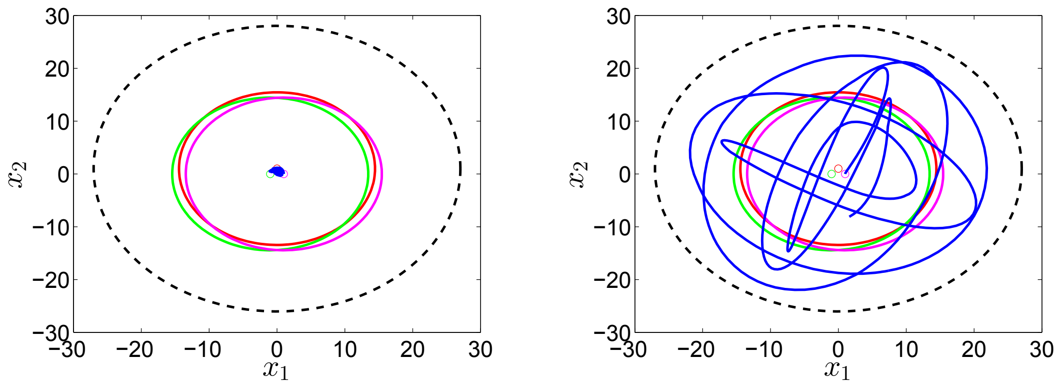
As in [12], we choose $V_1(x) = x_1^2 + (x_2 - 1)^2$, $V_2(x) = (x_1 + 1)^2 + x_2^2$, and $V_3(x) = (x_1 - 1)^2 + x_2^2$ as Lyapunov functions for subsystems 1, 2, and 3, respectively. The uniform convergence rate for these subsystems is $\lambda = 2$. The safe set $\underline{\mathcal{B}}$ is chosen as the sub-level set $\underline{\mathcal{B}} := \{x \in \mathbb{R}^2 \mid V_1(x) \leq 730\}$ and is denoted by the dashed black ellipse in Figs. 3(a) and 3(b). With $\kappa = 10$, we obtain an upper bound for $\mu(\kappa)$ and $\omega(\kappa)$ using [16, Proposition 1] as 2.66 and 27, respectively. Choosing $\overline{N}_0 = 1.1$ ensures that $\mathcal{M}(\mu(\kappa)^{1+\overline{N}_0}\omega(\kappa)) \subset \mathring{\underline{\mathcal{B}}}$, as required by (13); see Fig. 3(a) where $\mathcal{M}(\mu(\kappa)^{1+\overline{N}_0}\omega(\kappa))$ is denoted by the union of the red, green, and magenta ellipses. Furthermore, using (14) with $\epsilon = 0.01$, we obtain $\overline{N}_{\mathrm{a}} = 0.49$ s. We generate a random switching sequence that satisfies (7) with $N_0 = 1.1$ and $N_{\mathrm{a}} = 0.49$ and implement it on the switched system with $x(0) = x_3^* \in \bigcap_{p \in \mathcal{P}} \mathcal{M}_p(\omega(\kappa))$. It can be seen in Fig. 3(a) that the solution, denoted by blue, remains within $\mathcal{M}(\mu(\kappa)^{1+\overline{N}_0}\omega(\kappa))$. Now, we introduce a disturbance $d(t) = [25\sin(\sqrt{2}t), 25\sin(\sqrt{3}t)]^{\mathrm{T}}$ and continue switching according to the average dwell time bound using $N_0 = 1.1$ and $N_{\mathrm{a}} = 0.49$. The evolution of the state in this case is provided in Fig. 3(b), where it is seen that the solutions remain within $\mathring{\underline{\mathcal{B}}}$. Similarly, solutions with smaller disturbances also remained within $\mathring{\underline{\mathcal{B}}}$, as expected from Theorem 1.

VII. Conclusion

This paper presented a robustness result for switching among systems with multiple equilibria under disturbances using the notion of practical stability. It was shown that the average dwell-time bound for the switching signals obtained in the absence of disturbances, ensures safety of the switched system under sufficiently small disturbances. The hypotheses of Theorem 1 do not require knowledge of disturbances, thereby, facilitating the design of robust switching policies in situations where the disturbances are not known a priori. Although, our motivation for these results arises from motion planning of dynamic robots [8], [23], they are relevant to a much broader class of applications.

Appendix

Proof: [Claim] We will prove this claim by contradiction. Assume *ad absurdum* that $\mathcal{M}(\bar{\omega}(\delta)) \subset \mathring{\underline{\mathcal{B}}}$ and that solutions $x(t)$ starting at $x(0) \in \bigcap_{p \in \mathcal{P}} \mathcal{M}_p(\omega(\kappa))$ and evolving under $d \in \mathcal{D}$ with $\|d\|_\infty < \delta$ must leave $\mathring{\underline{\mathcal{B}}}$ in finite time $T > 0$. First, note that for any $d \in \mathcal{D}$ such that $\|d\|_\infty < \delta$, $\bar{\omega}(\|d\|_\infty) < \bar{\omega}(\delta)$ from (16). Hence, $\mathcal{M}(\bar{\omega}(\|d\|_\infty)) \subset \mathcal{M}(\bar{\omega}(\delta))$, which further leads to $\mathcal{M}(\bar{\omega}(\|d\|_\infty)) \subset \mathring{\underline{\mathcal{B}}}$ by



(a) (b)

Fig. 3. Switching among members of the family (29) with switching signals obtained from Theorem 1. The safe comapct set $\underline{\mathcal{B}}$ is denoted by black dashed ellipse, the red, green, and magenta ellipses are $\mathcal{M}_1(\mu(\kappa)^{1+\overline{N}_0}\omega(\kappa))$, $\mathcal{M}_2(\mu(\kappa)^{1+\overline{N}_0}\omega(\kappa))$, and $\mathcal{M}_3(\mu(\kappa)^{1+\overline{N}_0}\omega(\kappa))$, respectively. **(a)** Switching in the absence of disturbances. **(b)** Switching in the presence of the disturbance $d(t) = [25\sin(\sqrt{2}t), 25\sin(\sqrt{3}t)]^{\mathrm{T}}$.

the assumption of the claim. Now, by (23) we have that $x(t) \in \mathcal{M}(\bar{\omega}(\|d\|_\infty))$ for all $t \in [0, T)$ and since $x(t)$ is continuous we have that $x(T) = \lim_{t \nearrow T} x(t)$ is well defined. Furthermore, since $\mathcal{M}(\bar{\omega}(\|d\|_\infty))$ is compact, we have $x(T) \in \mathcal{M}(\bar{\omega}(\|d\|_\infty))$ so that the solution $x(t) \in \mathcal{M}(\bar{\omega}(\|d\|_\infty))$ over the closed interval $[0, T]$. On the other hand, since $\mathcal{M}(\bar{\omega}(\|d\|_\infty)) \subset \mathring{\underline{\mathcal{B}}}$ we also have that $x(T)$ must be in the open set $\mathring{\underline{\mathcal{B}}}$. Invoking continuity of $x(t)$ again, there must exist a $\delta_T > 0$ such that $x(t) \in \mathring{\underline{\mathcal{B}}}$ for all $T < t < T + \delta_T$, which implies that T cannot be the greatest lower bound on the exit time, leading to a contradiction with the definition of T in (22). As a result, $\mathcal{M}(\bar{\omega}(\|d\|_\infty)) \subset \mathring{\underline{\mathcal{B}}}$ leads to $T \to \infty$ and thus (23) holds for all $t \geq 0$. ■

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